# Projective Structure and Holonomy in 4-dimensional Lorentz Manifolds

Graham S.Hall[a], David P.Lonie[b],[*]

[a]*Institute of Mathematics, University of Aberdeen, Aberdeen, AB24 3UE, U.K.*
[b]*108e Anderson Drive, Aberdeen AB15 6BW, Scotland, UK*

## Abstract

This paper studies the situation when two 4-dimensional Lorentz manifolds (that is, space-times) admit the same (unparametrised) geodesics, that is, when they are projectively related. A review of some known results is given and then the problem is considered further by treating each holonomy type in turn for the space-time connection. It transpires that all holonomy possibilities can be dealt with completely except the most general one and that the consequences of two space-times being projectively related leads, in many cases, to their associated Levi-Civita connections being identical.

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## 1. Introduction and Notation

There has been some recent interest in the study of projective relatedness of (metric) connections and, in particular, within Einstein's theory [1, 2, 3, 4, 5, 6, 7, 8, 9]. Thus, roughly speaking, one assumes that one has two Lorentz metrics on a given space-time $M$ whose Levi-Civita connections give rise to the same set of geodesic paths (*unparametrised* geodesics) on the space-time manifold and then tries to find the relationship between these metrics and connections. Such a problem has an immediate geometrical appeal but is also of obvious interest within Einstein's general relativity because of the Newton-Einstein principle of equivalence. This paper continues the general study of finding those metrics $g$ whose Levi-Civita connection $\nabla$ can be recovered uniquely from its geodesics, that is, $\nabla$ is such that if any other Levi-Civita connection $\nabla'$ of a Lorentz metric $g'$ on $M$ has the same geodesic paths as those of $\nabla$ then $\nabla = \nabla'$. If this is the case, the relationship between $g$ and $g'$ can be found from holonomy theory. If $\nabla = \nabla'$ is not the conclusion one can still compute $g'$ in terms of $g$ in all cases except the most general holonomy case and even for this situation the solution for $g'$ can be achieved under certain conditions. A brief summary of some results in this direction is given in sections 4 and 5 following preliminary work in the next two sections. Sections 5 and 6 of this paper will concentrate on a study which reveals a close relationship between the holonomy type of a space-time and the varying degrees of ability of its associated Levi-Civita connection to be recoverable from its geodesic paths.

Throughout this paper, $M$ will denote a 4-dimensional, Hausdorff, connected, smooth manifold which admits a smooth metric $g$ of Lorentz signature $(-, +, +, +)$. The pair $(M, g)$ is called a *space-time*. All structures naturally occurring on $M$ will be assumed smooth. The unique, symmetric Levi-Civita connection arising on $M$ through $g$ is denoted by $\nabla$ and, in a coordinate domain of $M$, its Christoffel symbols are written $\Gamma^a_{bc}$. The type $(1, 3)$ curvature tensor associated with $\nabla$ is denoted by $Riem$ and its (coordinate) components are written $R^a{}_{bcd}$. The Ricci tensor, $Ricc$, derived from $Riem$, has components $R_{ab} \equiv R^c{}_{acb}$ and $R \equiv R_{ab}g^{ab}$ is the Ricci scalar.

[*]Corresponding author
*Email addresses:* `g.hall@abdn.ac.uk` (Graham S.Hall), `DLonie@aol.com` (David P.Lonie)

For $m \in M$, $T_m M$ denotes the *tangent space to $M$ at $m$* and this will, for convenience, be identified with the cotangent space to $M$ at $m$, through the isomorphism arising from the metric $g(m)$ by index raising and lowering. If no ambiguity could arise, the same symbol will be used for both a vector (field) and its associated covector (field). A tetrad $u, x, y, z \in T_m M$ is called *orthonormal* if and only if the only non-vanishing inner products between tetrad members are $-g(u,u) = g(x,x) = g(y,y) = g(z,z) = 1$ and a tetrad $l, n, x, y \in T_m M$ is called *null* if and only if the only non-vanishing inner products between tetrad members are $g(l,n) = g(x,x) = g(y,y) = 1$. In this case, $l$ and $n$ are null vectors. A space-time will be called *non-flat* if $Riem$ does not vanish over any non-empty open subset of $M$. This is a physical requirement sometimes imposed to prevent gravitational shielding in Einstein's theory. Round and square brackets around indices denote the usual symmetrisation and skew-symmetrisation of those indices. Let $m \in M$ and let $\Omega_m$ denote the 6-dimensional vector space of all skew-symmetric tensors at $m$, irrespective of tensor type (that is, of the positions of the indices) because of the isomorphisms resulting from the metric $g(m)$. Members of any of these vector spaces are, somewhat loosely, called *bivectors*. If $F \in \Omega_m$ with components $F^{ab}$ then since $F^{ab}$ is skew-symmetric, its matrix rank is even. If this rank is two, $F$ is called *simple* and may be written as $F^{ab} = 2p^{[a}q^{b]}$ for $p, q \in T_m M$. The 2-dimensional subspace (2-space) of $T_m M$ spanned by $p$ and $q$ is uniquely determined by $F$ and called the *blade* of $F$. This bivector (or its blade) will sometimes be denoted by $p \wedge q$. A simple bivector $F$ at $m$ is then called *timelike*, *spacelike* or *null* according as its blade is, respectively, a timelike, spacelike or null 2-space at $m$. If $F \in \Lambda_m M$ has rank four it is called *non-simple* and may be written as $F = G + H$ where $G$ and $H$ are simple bivectors with $G$ timelike and $H$ spacelike and where the blades of $G$ and $H$ are uniquely determined by $F$ and are orthogonal complements of each other. The union of these blades spans $T_m M$. They will be collectively called the *canonical pair of blades* of $F$. The symbol $*$ will denote the usual duality operator; thus $G$ and $H$ above are duals of each other (up to scalings).

A remark is added here about the notation in this paper. Both a coordinate and an index-free notation will be used. However, since many of the calculations involve tensor contractions and a coordinate approach is generally much shorter for these, the former will naturally predominate here.

## 2. Curvature structure of space-times

Because of the algebraic symmetries of $Riem$, one may introduce the *curvature map $f$* from the vector space of bivectors to the space of type $(1,1)$ (skew-self adjoint with respect to $g(m)$) tensors at $m$ (and with the latter space identified with the space of bivectors at $m$ and referred to as such) by

$$f : F^{ab} \to R^a{}_{bcd} F^{cd} \tag{1}$$

The rank of the linear map $f$ is referred to as the *curvature rank (of Riem)* at $m$ [10]. It is clear that if $F$ is a bivector which is defined and smooth on some open subset of $M$ then so also is $f(F)$ where $f(F)(m) = f(F(m))$. Let $B_m$ denote the range space of $f$ at $m$ (suitably interpreted according to the agreed identification of bivectors) so that $\dim B_m$ equals the curvature rank at $m$ and is $\leq 6$. This leads to a convenient algebraic classification of $Riem$ at $m$ into five mutually exclusive and disjoint *curvature classes* (for further details, see [10]).

**Class A** This covers all possibilities not covered by classes **B**, **C**, **D** and **O** below. For this class, the curvature rank at $m$ is 2, 3, 4, 5 or 6.

**Class B** This occurs when $\dim B_m = 2$ and when $B_m$ is spanned by a timelike-spacelike pair of simple bivectors with orthogonal blades (chosen so that one is the dual of the other). In this case, one can choose a null tetrad $l, n, x, y \in T_m M$ such that these bivectors are $F = l \wedge n$ and $\overset{*}{F} = x \wedge y$ so that $F$ is timelike and $\overset{*}{F}$ is spacelike and then (using the algebraic identity $R_{a[bcd]} = 0$ to remove cross terms) one has, at $m$,

$$R_{abcd} = \alpha F_{ab} F_{cd} + \beta \overset{*}{F}_{ab} \overset{*}{F}_{cd} \tag{2}$$

for $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0 \neq \beta$.

**Class C** In this case $\dim B_m = 2$ or $3$ and $B_m$ may be spanned by independent simple bivectors $F$ and $G$ (or $F$, $G$ and $H$) with the property that there exists $0 \neq r \in T_m M$ such that $r$ lies in the blades of $\overset{*}{F}$ and $\overset{*}{G}$ (or $\overset{*}{F}$, $\overset{*}{G}$ and $\overset{*}{H}$). Thus $F_{ab}r^b = G_{ab}r^b(= H_{ab}r^b) = 0$ and $r$ is then unique up to a multiplicative non-zero real number.

**Class D** In this case $\dim B_m = 1$. If $B_m$ is spanned by the bivector $F$ then, at $m$,

$$R_{abcd} = \alpha F_{ab}F_{cd} \tag{3}$$

for $0 \neq \alpha \in \mathbb{R}$ and $R_{a[bcd]} = 0$ implies that $F_{a[b}F_{cd]} = 0$ from which it may be checked that $F$ is necessarily simple.

**Class O** In this case $Riem$ vanishes at $m$.

This classification is, of course, pointwise and may vary over $M$. A space-time $(M, g)$ (or some subset of it) which has the same curvature class at each point of $M$ is said to be *of that class*. The subset of $M$ consisting of points at which the curvature class is **A** is an open subset of $M$ ([10], p 393) and the analogous subset arising from the class **O** is closed (and has empty interior in the manifold topology of $M$ if $(M, g)$ is non-flat). It is important in what is to follow that *the equation $R_{abcd}k^d = 0$ at $m$ has no non-trivial solutions for $k \in T_m M$ if the curvature class at $m$ is **A** or **B**, a unique independent solution (the vector $r$ above) if the curvature class at $m$ is **C** and two independent solutions if the curvature class at $m$ is **D** (and which span the blade of $\overset{*}{F}$ in (3)). If $\dim B_m \geq 4$ the curvature class at $m$ is necessarily **A**.*

The following result will be useful in what is to follow and the details and proof can be found in [10].

**Theorem 1.** *Let $(M, g)$ be a space-time, let $m \in M$ and let $h$ be a non-zero second order, symmetric, type (0,2) (not necessarily non-degenerate) tensor at $m$ satisfying $h_{ae}R^e{}_{bcd} + h_{be}R^e{}_{acd} = 0$.*

*(i) If the curvature class of $(M, g)$ at $m$ is **D** and $u, v \in T_m M$ span the 2-space at $m$ orthogonal to the blade of $F$ in (3) (that is $u \wedge v$ is the blade of $\overset{*}{F}$), there exists $\phi, \mu, \nu, \lambda \in \mathbb{R}$ such that, at $m$,*

$$h_{ab} = \phi g_{ab} + \mu u_a u_b + \nu v_a v_b + \lambda(u_a v_b + v_a u_b) \tag{4}$$

*(ii) If the curvature class of $(M, g)$ at $m$ is **C** there exists $r \in T_m M$ (the vector appearing in the above definition of class **C**) and $\phi, \lambda \in \mathbb{R}$ such that, at $m$,*

$$h_{ab} = \phi g_{ab} + \lambda r_a r_b \tag{5}$$

*(iii) If the curvature class of $(M, g)$ at $m$ is **B** there exists a null tetrad $l, n, x, y$ (that appearing in the above definition of class **B**) and $\phi, \lambda \in \mathbb{R}$ such that, at $m$ (and making use of the associated completeness relation),*

$$h_{ab} = \phi g_{ab} + \lambda(l_a n_b + n_a l_b) = (\phi + \lambda)g_{ab} - \lambda(x_a x_b + y_a y_b) \tag{6}$$

*(iv) If the curvature class of $(M, g)$ at $m$ is **A** there exists $\phi \in \mathbb{R}$ such that, at $m$,*

$$h_{ab} = \phi g_{ab} \tag{7}$$

The proof of theorem 1 is based on the fact that the members of the range $B_m$ of the map $f$ in (1) are, as type $(1, 1)$ tensors, skew-self adjoint with respect to $g$, and also skew-self adjoint with respect to $h$. Thus each $F \in B_m$ satisfies

$$g_{ac}F^c{}_b + g_{bc}F^c{}_a = 0 \qquad h_{ac}F^c{}_b + h_{bc}F^c{}_a = 0 \tag{8}$$

It is a consequence of (8) that the blade of $F$ (if $F$ is simple) and each of the canonical pair of blades of $F$ (if $F$ is non-simple) are eigenspaces of $h$ with respect to $g$, that is, for $F$ simple, any $k \in T_m M$ in the blade

of $F$ satisfies $h_{ab}k^b = \omega g_{ab}k^b$ where the eigenvalue $\omega \in \mathbb{R}$ is independent of $k$, and similarly for each of the canonical blades if $F$ is non-simple (but with possibly different eigenvalues for these blades) [10].

[It is remarked that if $\tilde{g}$ is another metric on $M$ whose curvature tensor $\tilde{Riem}$ equals the curvature tensor $Riem$ of $g$ *everywhere on $M$* then the conditions of this theorem are satisfied for $h = \tilde{g}(m)$ at each $m \in M$ and so the conclusions also hold except that now one must add the restriction $\phi \neq 0$ in each case to preserve the non-degeneracy of $\tilde{g}$ at $m$ and maybe some restrictions on $\phi, \mu, \nu$ and $\lambda$ if the signature of $\tilde{g}$ is prescribed. If $(M, g)$ is of curvature class $\mathbf{A}$, (7) gives $\tilde{g} = \phi g$ for some (smooth) function $\phi$ on $M$ and the Bianchi identity may be used to show that $\phi$ is constant on $M$ [10]. Further, if the condition on $g$ and $\tilde{g}$ of having equal curvatures is weakened to $\tilde{g}_{ab;[cd]} = 0$ on $M$, where a semi-colon denotes a $\nabla$-covariant derivative, the Ricci identity for $\tilde{g}$ shows that the condition of theorem 1 is satisfied for $\tilde{g} = h$ on $M$. In particular, from part (iv) above, if $(M, g)$ is of curvature class $\mathbf{A}$ the condition that $g$ and $\tilde{g}$ are conformally related is equivalent to $\tilde{g}_{ab;[cd]} = 0$.]

## 3. Holonomy theory

Let $m \in M$ and for $1 \leq k \leq \infty$ let $C_k(m)$ denote the set of all piecewise $C^k$ closed curves starting and ending at $m$. If $c \in C_k(m)$ let $\tau_c$ denote the vector space isomorphism of $T_mM$ obtained by parallel transporting, using $\nabla$, each member of $T_mM$ along $c$. Using a standard notation associated with curves one defines, for curves $c, c_0, c_1, c_2 \in C_k(m)$, with $c_0$ denoting a constant curve at $m$, the *identity map* $\tau_{c_0}$ on $T_mM$, the *inverse* $\tau_c^{-1} \equiv \tau_{c^{-1}}$ and *product* $\tau_{c_1} \cdot \tau_{c_2} \equiv \tau_{c_1 \cdot c_2}$ to put a group structure on $C_k(m)$, making it a subgroup of $G \equiv GL(T_mM)(= GL(4, \mathbb{R}))$, called the $k-$*holonomy group* of $M$ at $m$ and denoted by $\Phi_k(m)$. In fact, since $M$ is connected and also a manifold, it is also path connected and, as a consequence, it is easily checked that, up to an isomorphism, $\Phi_k(m)$ is independent of $m$. Less obvious is the fact that $\Phi_k(m)$ is independent of $k$ ($1 \leq k \leq \infty$) and thus one arrives at the *holonomy group* $\Phi$ (of $\nabla$) on $M$. It can now be proved that $\Phi$ is a Lie group which is connected if $M$ is simply connected. The Lie algebra $\phi$ of $\Phi$ is called the *holonomy algebra*.

If the above operations are repeated this time using only curves in $C_k(m)$ homotopic to zero one similarly achieves the *restricted holonomy group* $\Phi^0$ of $\nabla$ on $M$ and which can be shown to be the identity component of $\Phi$ and is equal to $\Phi$ if $M$ is simply connected. The connection $\nabla$ can be shown to be *flat* (that is, $Riem$ vanishes on $M$) if and only if $\phi$ is trivial. Since, for a space-time $(M, g)$, $\nabla$ is compatible with the metric $g$, that is, $\nabla g = 0$, each map $\tau_c$ on $T_mM$ preserves inner products with respect to $g(m)$. It can then be shown that $\Phi$ is (isomorphic to) a Lie subgroup of the *Lorentz group* $\mathcal{L}$. Thus the holonomy algebra $\phi$ can be identified with a subalgebra of the Lie algebra $L$ of $\mathcal{L}$, the *Lorentz algebra*. The one-to-one correspondence between the subalgebras of $L$ and the *connected* Lie subgroups of $\mathcal{L}_0$ shows that if $M$ is simply connected the (connected) Lie group $\Phi$ is determined by the subalgebra of $L$ associated with $\phi$. Further details may be found in [11] and a summary in [10].

The subalgebra structure of $L$ is well-known and can be conveniently represented by informally identifying it with certain Lie algebras of bivectors in a well-defined way. The binary operation on $L$ is then that of matrix commutation. Such a representation of $L$ is well-known and has been classified into fifteen convenient types [12] (for details of the possible holonomy types most relevant for the physics of general relativity see [13, 10]). This representation is given in the first three columns of table 1 using either a null tetrad $l, n, x, y$ or an orthonormal tetrad $u, x, y, z$ to describe a basis for each subalgebra. It is noted that in types $R_5$ and $R_{12}$, $0 \neq \omega \in \mathbb{R}$ (and, in fact, $R_5$ cannot occur as the holonomy algebra for a space-time, but each of the others can - see, e.g. [10, 14]). There is also a type $R_1$, when $\phi$ is trivial and $(M, g)$ is flat, but this trivial type is omitted. Type $R_{15}$ is the "general" type, when $\phi = L$.

If $M$ is simply connected, $\Phi$ is connected and is then uniquely determined by the subalgebra of $L$ associated with $\phi$. In any case, $\Phi$ will be referred to according to its Lie algebra label as in table 1. The simple connectedness condition will not necessarily be imposed on $M$; it is often sufficient to have this condition locally and a simply connected, connected chart domain $V$ is always available. Then the holonomy algebra of the restricted space-time $(V, g)$ is a subalgebra of $\phi$. However, if $M$ is simply connected, the (nowhere-zero) covariantly constant and recurrent vector fields (indicated in table 1 by enclosing a basis for the vector space of covariantly constant vector fields on $M$ for each holonomy type inside $<>$ brackets in the final column

4

Table 1: Holonomy algebras

| Type | Dim. | Basis | Curvature Class | Recurrent vector fields | Constant vector fields |
|------|------|-------|-----------------|-------------------------|------------------------|
| $R_2$ | 1 | $l \wedge n$ | $\mathbf{D}$ or $\mathbf{O}$ | $\{l\}, \{n\}$ | $< x, y >$ |
| $R_3$ | 1 | $l \wedge x$ | $\mathbf{D}$ or $\mathbf{O}$ | - | $< l, y >$ |
| $R_4$ | 1 | $x \wedge y$ | $\mathbf{D}$ or $\mathbf{O}$ | - | $< l, n >$ |
| $R_5$ | 1 | $l \wedge n + \omega(x \wedge y)$ | - | - | - |
| $R_6$ | 2 | $l \wedge n, l \wedge x$ | $\mathbf{C}, \mathbf{D}$ or $\mathbf{O}$ | $\{l\}$ | $< y >$ |
| $R_7$ | 2 | $l \wedge n, x \wedge y$ | $\mathbf{B}, \mathbf{D}$ or $\mathbf{O}$ | $\{l\}, \{n\}$ | - |
| $R_8$ | 2 | $l \wedge x, l \wedge y$ | $\mathbf{C}, \mathbf{D}$ or $\mathbf{O}$ | - | $< l >$ |
| $R_9$ | 3 | $l \wedge n, l \wedge x, l \wedge y$ | $\mathbf{A}, \mathbf{C}, \mathbf{D}$ or $\mathbf{O}$ | $\{l\}$ | - |
| $R_{10}$ | 3 | $l \wedge n, l \wedge x, n \wedge x$ | $\mathbf{C}, \mathbf{D}$ or $\mathbf{O}$ | - | $< y >$ |
| $R_{11}$ | 3 | $l \wedge x, l \wedge y, x \wedge y$ | $\mathbf{C}, \mathbf{D}$ or $\mathbf{O}$ | - | $< l >$ |
| $R_{12}$ | 3 | $l \wedge x, l \wedge y, l \wedge n + \omega(x \wedge y)$ | $\mathbf{A}, \mathbf{C}, \mathbf{D}$ or $\mathbf{O}$ | $\{l\}$ | - |
| $R_{13}$ | 3 | $x \wedge y, y \wedge z, x \wedge z$ | $\mathbf{C}, \mathbf{D}$ or $\mathbf{O}$ | - | $< u >$ |
| $R_{14}$ | 4 | $l \wedge n, l \wedge x, l \wedge y, x \wedge y$ | any | $\{l\}$ | - |
| $R_{15}$ | 6 | $L$ | any | - | - |

and by listing the independent *properly recurrent* vector fields for each holonomy type in { } brackets in the second from last column) can be taken as globally defined on $M$. If $M$ is not simply connected each $m \in M$ admits a simply connected and connected open neighbourhood $V$ on which these (nowhere-zero) covariantly constant or recurrent vector fields are defined but the holonomy type of the restricted space-time $(V, g)$ may differ from that of $(M, g)$. [A global nowhere-zero vector field $X$ on a connected, open subset $V$ of $M$ is called *recurrent* on $V$ if it satisfies $\nabla X = X \otimes w$ for some global, smooth covector field $w$ on $V$ (the *recurrence 1-form*). In this case, the *sign* (including zero) of $g(X, X)$ is constant on $V$ and clearly every non-trivial covariantly constant vector field on $V$ is recurrent on $V$. Then $X$ is called *properly recurrent* on $V$ if it is recurrent on $V$ and is such that no function $\alpha : V \to \mathbb{R}$ exists such that $\alpha$ is nowhere zero on $V$ and $\alpha X$ is covariantly constant on $V$. In fact, *any non-null* recurrent vector field on $(M, g)$ with the latter of arbitrary signature and dimension, or *any* recurrent vector field on a manifold of arbitrary dimension and with positive definite metric can be globally scaled to be nowhere zero and covariantly constant because if $Y$ is any such vector field and $\nabla Y = Y \otimes r$ for some 1-form $r$, $\alpha Y$ is covariantly constant, where $\alpha = |(g(Y, Y)|^{-\frac{1}{2}}$. Thus properly recurrent vector fields on $V$ are null everywhere on $V$.] A study of the vector space of *global, covariantly constant, type* $(0, 2)$ *symmetric tensor fields* (or local such tensor fields on a fixed open subset of $M$) for each holonomy type can be obtained from the above theory and full details can be found in [10, 15]. It will be of importance later. In particular, for the holonomy types $R_9$, $R_{12}$, $R_{14}$ and $R_{15}$, the only such global tensors are constant multiples of the metric tensor $g$.

A recurrent vector field spans a 1-dimensional distribution on its domain of definition which is preserved by parallel transport. There is an important generalisation of this concept. Let $m \in M$ and $W$ a non-trivial proper subspace of $T_m M$. Suppose $\tau_c(W) = W$ for each $\tau_c$ arising from $c \in C_k(m)$ at $m$. Then $W$ is *holonomy invariant* and gives rise in an obvious way, by parallel transport, to a smooth distribution on $M$ which is, in fact, integrable [11]. Clearly, if $W \subset T_m M$ is holonomy invariant then so is the orthogonal complement, $W^\perp$, of $W$. If such a $W$ exists the holonomy group $\Phi$ of $M$ is called *reducible* (otherwise, *irreducible*). More details can be found in [11, 16] (see also [10, 17]). Thus, for example, in table 1 the holonomy type $R_2$ admits, locally, two 1-dimensional, null holonomy invariant subspaces spanned by $l$ and $n$ and which give rise, locally, to two null, properly recurrent vector fields and infinitely many 1-dimensional spacelike holonomy invariant subspaces spanned by the infinitely many covariantly constant vector fields in $< x, y >$. For the holonomy type $R_7$, two 1-dimensional null holonomy invariant subspaces exist locally and which give rise, locally, to two independent properly recurrent null vector fields as in the previous case, together with a 2-dimensional spacelike one orthogonal to each of the null ones. For holonomy types $R_{10}$, $R_{11}$ and $R_{13}$ one

has a local 1-dimensional holonomy invariant subspace, spanned by a covariantly constant vector field in each case, together with its (3-dimensional) orthogonal complement. These holonomy decompositions will be useful in what is to follow.

It is useful, at this point, to introduce the *infinitesimal holonomy group* $\Phi'_m$, of $(M, g)$ at any $m \in M$. Again using a semi-colon to denote a $\nabla-$covariant derivative, consider, in some coordinate neighbourhood of $m$, the following matrices for $(M, g)$ at $m$

$$R^a{}_{bcd}X^cY^d, \quad R^a{}_{bcd;e}X^cY^dZ^e, \quad \ldots \tag{9}$$

for $X, Y, Z, \ldots \in T_mM$. It turns out that the collection (9) spans a subalgebra of the holonomy algebra $\phi$ (and hence only a finite number of independent terms arise in (9) [11]). This algebra is called the *infinitesimal holonomy algebra at m* and is denoted by $\phi'_m$. The unique connected Lie subgroup of $\Phi$ that it gives rise to is the *infinitesimal holonomy group* $\Phi'_m$ at $m$. Thus the range space of the map $f$ in (1) is, at each $m \in M$, (isomorphic as a vector space to) a subspace of $\phi$. This gives a restriction, when $\phi$ is known, on $f$, the expression for $Riem$ at each $m$ and also on the curvature class of $Riem$ at $m$. This last restriction is listed in the fourth column of table 1. In addition, it is remarked here for later convenience that if $(M, g)$ has holonomy type $R_{14}$ it cannot be of curvature class **B** everywhere (see the comments before theorem 4). A useful relationship between the various algebras $\phi'_m$, the algebra $\phi$ and the curvature class (through the range space $B_m$) at each $m \in M$ is provided by the Ambrose-Singer theorem [18] (see also [11]). This theorem says, roughly speaking, that the holonomy algebra of $(M, g)$ can be constructed by first choosing (any) $m \in M$ and, for each $m' \in M$ and each piecewise $C^1$ curve $c$ from $m'$ to $m$, computing the range space of $f$ at $m'$ and parallely transporting each member of it to $m$ along $c$. If this is done for each such $m'$ and $c$, the collection of bivectors accumulated at $m$ spans the holonomy algebra of $(M, g)$.

## 4. Projective structure

In general relativity the study of projective structure is motivated by the Newton-Einstein principle of equivalence. Consider the following question; for a space-time $(M, g)$ with Levi-Civita connection $\nabla$, if one knows the paths of all the *unparameterised* geodesics (that is, only the *geodesic paths* in $M$ without regard to their parameters) how tightly are $\nabla$ and $g$ determined? Suppose that $(M, g)$ is a space-time and $g'$ is another metric on $M$ of arbitrary signature, with respective Levi-Civita connections $\nabla$ and $\nabla'$, such that the sets of geodesic paths of $\nabla$ and $\nabla'$ coincide. Then $\nabla$ and $\nabla'$ (or $g$ and $g'$, or $(M, g)$ and $(M, g')$) are said to be *projectively related* (on $M$)). [In fact, it is sufficient that, for each $m \in M$, $\nabla$ and $\nabla'$ share a non-empty subset of unparametrised $g$-timelike geodesics whose directions at $m$ span a non-empty open subset in the usual topology on the collection of 1-dimensional subspaces (directions) of $T_mM$ [3].] Although, in general, a projectively related pair $\nabla$ and $\nabla'$ may still be expected to differ, it turns out that in many interesting situations they are necessarily equal. Intuitively, one might expect a link between projective relatedness and holonomy type and if $\nabla = \nabla'$ is the result, $(M, g)$ and $(M, g')$ have the same holonomy type. Holonomy theory can then describe precisely the (simple) relationship between $g$ and $g'$ (and the signatures of $g$ and $g'$ may differ) [15, 10]. If $\nabla = \nabla'$, $g$ and $g'$ will be referred to as *trivially projectively related* (or *affinely related*). For later convenience in the construction of examples, the concept of *local projective relatedness* will be required. For a space-time $(M, g)$ let $U$ be a non-empty connected open subset of $M$ and let $g'$ be some metric defined on $U$. Then $g$ and $g'$ (or their respective Levi-Civita connections, etc) will be said to be *(locally) projectively related* (on $U$) if (the restriction of) $g$ on $U$ is projectively related to $g'$ on $U$. [Generally speaking, this paper is concerned with the situation when $g'$ is also of Lorentz signature but this will not be assumed.]

If $\nabla$ and $\nabla'$ are projectively related then there exists a uniquely defined, global, smooth 1-form field $\psi$ on $M$ such that, in any coordinate domain of $M$, the respective Christoffel symbols of $\nabla$ and $\nabla'$ satisfy [19, 20, 21]

$$\Gamma'^a_{bc} - \Gamma^a_{bc} = \delta^a_b \psi_c + \delta^a_c \psi_b \tag{10}$$

Conversely, if this condition is satisfied for some global 1-form $\psi$ on $M$, $\nabla$ and $\nabla'$ are projectively related. It is a consequence of the fact that $\nabla$ and $\nabla'$ are *metric connections* that $\psi$ is a *global* gradient on $M$ (see,

e.g. [21]). Equation (10) can, by using the identity $\nabla' g' = 0$, be written in the equivalent form

$$g'_{ab;c} = 2g'_{ab}\psi_c + g'_{ac}\psi_b + g'_{bc}\psi_a \tag{11}$$

Equation (10) reveals a relation between the type $(1,3)$ curvature tensors $Riem$ and $Riem'$ of $\nabla$ and $\nabla'$, respectively, given by

$$R'^a{}_{bcd} = R^a{}_{bcd} + \delta^a{}_d\psi_{bc} - \delta^a{}_c\psi_{bd} \qquad (\Rightarrow R'_{ab} = R_{ab} - 3\psi_{ab}) \tag{12}$$

where $\psi_{ab} \equiv \psi_{a;b} - \psi_a\psi_b = \psi_{ba}$ and where $R'_{ab} \equiv R'^c{}_{acb}$ are the Ricci tensor components of $\nabla'$.

A particularly important case of such a study arises where the original pair $(M, g)$ is a space-time which is also an *Einstein space* so that the Ricci and metric tensors are related by $Ricc = \frac{R}{4}g$. This problem has been discussed in several places (see the bibliography in [6]). The particular case which is, perhaps, of most importance in general relativity arises when the Ricci scalar vanishes and then $(M, g)$ is a *vacuum* (*Ricci-flat*) space-time and this is discussed in [1, 3, 5, 8]. It turns out that if $(M, g)$ is a space-time which is an Einstein space and if $g'$ is another metric on $M$ projectively related to $g$, either $(M, g)$ and $(M, g')$ are each of constant curvature, or the Levi-Civita connections $\nabla$ and $\nabla'$ of $g$ and $g'$, respectively, are equal. Since if one of $(M, g)$ and $(M, g')$ is of constant curvature, so is the other [21], then if $(M, g)$ is not of constant curvature, (and so $\nabla' = \nabla$) an argument from holonomy theory can be used to show that, *generically*, $(M, g')$ is also an Einstein space *and* $g' = cg$ ($0 \neq c \in \mathbb{R}$). Although examples exist where each of these conclusions fail, $g'$ always has Lorentz signature (up to an overall minus sign). If, in addition, $(M, g)$ is assumed vacuum and the non-flat condition is imposed on it, then necessarily $\nabla = \nabla'$ and, with one very special case excluded (the pp-waves!), $g' = cg$ on $M$ ($0 \neq c \in \mathbb{R}$) (and so $(M, g')$ is also vacuum). For this case $g'$ has the same signature as $g$ (up to an overall minus sign [5]). This result is important for describing the power of the equivalence principle in general relativity theory. A similar restrictive result for space-times of certain holonomy types which are not Einstein spaces will be established in the next two sections.

With $(M, g)$ given, the object is to find all pairs $(g', \psi)$ satisfying (11) where, it is recalled, $\psi$ is a global gradient, $\psi = d\chi$, for some smooth function $\chi: M \to \mathbb{R}$. It is, however, convenient to replace the pair $(g', \psi)$ by the pair $(a, \lambda)$ where $a$ is a global type $(0,2)$ symmetric tensor field and $\lambda$ a global 1-form field on $M$ and which are related to the previous pair by the Sinjukov transformation [2]

$$a_{ab} = e^{2\chi}g'^{cd}g_{ac}g_{bd} \qquad \lambda_a = -e^{2\chi}\psi_b g'^{bc}g_{ac} \; (\Rightarrow \lambda_a = -a_{ab}\psi^b) \tag{13}$$

where a temporary abuse of notation has been used in that $g'^{ab}$ denotes the contravariant components of $g'$ (and not the tensor $g'_{ab}$ with indices raised using $g$) so that $g'_{ac}g'^{cb} = \delta_a{}^b$. Then (13) may be inverted to give

$$g'^{ab} = e^{-2\chi}a_{cd}g^{ac}g^{bd} \qquad \psi_a = -e^{-2\chi}\lambda_b g^{bc}g'_{ac} \tag{14}$$

Given that $g$ and $g'$ are projectively related metrics on $M$, so that (11) holds for some 1-form $\psi(= d\chi)$, then $a$ and $\lambda$ as defined in (13) can be shown, after a short calculation, to satisfy Sinjukov's equation

$$a_{ab;c} = g_{ac}\lambda_b + g_{bc}\lambda_a \tag{15}$$

Now (13) implies that $a$ is non-degenerate. It is, in fact, of the same signature as $g'$, this being easily checked by noting that at each $m \in M$ the map $T_mM \to T_mM$ given by $k^a \to k'^a = e^\chi g'^{ab}g_{bc}k^c$ is a vector space isomorphism and an isometry between the (generalised) inner product spaces $(T_mM, a(m))$ and $(T_mM, g'(m))$ since $g'_{ab}k'^a k'^b = a_{ab}k^a k^b$. A contraction of (15) with $g^{ab}$ then shows that $\lambda$ is a global gradient on $M$ (in fact, of $\frac{1}{2}a_{ab}g^{ab}$). It is remarked for future reference that if the right hand side of (15) vanishes at any $m \in M$ so also does $\lambda(m)$ (as a simple contraction of (15) with $g^{ac}$ shows). In summary, the vanishing of $\lambda$ on $M$ is equivalent to the vanishing of $\psi$ on $M$ and also to the statement that $\nabla = \nabla'$ on $M$.

In practice, to determine which pairs $(g', \nabla')$ are projectively related to some original pair $(g, \nabla)$ on $M$, it is often easier to use (15) to attempt to find $a$ and $\lambda$ rather than (11) to find $g'$ and $\psi$. This then requires converting back from $a$ and $\lambda$ to $g'$ and $\psi$. For this purpose the following lemma (the proof of which may be of interest in its own right) is useful. In it, the symmetric non-degenerate type $(2, 0)$ tensor $a^{-1}$ on $M$ is,

for each $m \in M$, the inverse of $a$ $(a_{ac}(a^{-1})^{cb} = \delta_a{}^b)$. Then raising and lowering indices on $a$ and $a^{-1}$ with $g$ in the usual way, one has a global type $(0,2)$ tensor (also labelled $a^{-1}$) with components $a_{ab}^{-1} = a^{-1cd}g_{ac}g_{bd}$ (and so $a_{ac}^{-1}a^{cb} = \delta_a{}^b$).

**Lemma 1.** *(cf [23])*

*Given a solution pair $(a, \lambda)$ of (15) for $(M, g)$ one may associate with it a unique solution pair $(g', \psi)$ of (11) by the following procedure. First construct the function $F : M \to \mathbb{R}$ by $F = |\det g|/|\det a|$ and then define a type $(0,2)$ metric $g'$ by $g' = Fa^{-1}$. Then define a function $\chi : M \to \mathbb{R}$ by $\chi = \frac{1}{2}\ln F$ and finally define a 1-form $\psi$ by $\psi = d\chi$. The pair $(g', \psi)$ is then a solution to (11) and together with the original pair $(a, \lambda)$, satisfies (13) and (14).*

**Proof**

Define a connection $\tilde{\nabla}$ on $M$ by its Christoffel symbols $\tilde{\Gamma}_{bc}^a$ in any coordinate system which are given in terms of the Christoffel symbols of $\nabla$ by $\tilde{\Gamma}_{bc}^a = \Gamma_{bc}^a + g_{bc}(a^{-1})^{ad}\lambda_d$. Then (15) implies $\tilde{\nabla}a = 0$ and so $\tilde{\nabla}$, being symmetric, is the Levi-Civita connection for the tensor $a$ when the latter is taken as a metric on $M$. Now a standard result gives $\Gamma_{ab}^a = \frac{\partial}{\partial x^b}\left(\frac{1}{2}\ln|detg|\right)$ and so, applying this to $\tilde{\nabla}$

$$\frac{\partial}{\partial x^b}\left(\frac{1}{2}\ln|\det a|\right) = \frac{\partial}{\partial x^b}\left(\frac{1}{2}\ln|\det g|\right) + (a^{-1})_{bd}\lambda^d \tag{16}$$

Now define $\psi_b \equiv -(a^{-1})_{bd}\lambda^d$ so that $\psi = d\chi$ is a global gradient, $\chi \equiv \frac{1}{2}\ln(|\det g|/|\det a|)$. Then since $(a^{-1})_{ad}a^{de} = \delta_a^d$ a covariant differentiation and use of (15) gives [23]

$$(a^{-1})_{ab;c} = (a^{-1})_{bc}\psi_a + (a^{-1})_{ac}\psi_b \tag{17}$$

Then $g' \equiv Fa^{-1} = e^{2\chi}a^{-1}$ and $\psi$ can be shown to satisfy (11). It is straightforward to check that (13) and (14) hold and that $\psi_a = -a_{ab}^{-1}\lambda^b$ (and so $\lambda_a = -a_{ab}\psi^b$). $\square$

It follows that *all projectively related metrics $g'$ together with their associated 1-forms $\psi$ will be found if all pairs $(a, \lambda)$ can be found satisfying (15) and with $a$ non-degenerate.* [In the above definition of $g'(\equiv Fa^{-1})$ it is pointed out that, in [23], a typographical error arose which resulted in $g'$ being erroneously written as $g' = e^{2\chi}a(= Fa)$.]

Denote the collection of *all* solution pairs $(g', \psi)$ of (11) with $g'$ a Lorentz metric on $M$, as described above by $Q'$ and let $P'$ denote the collection of *all* pairs $(a, \lambda)$ where $a$ is a type $(0,2)$ non-degenerate tensor and $\lambda$ a global 1-form on $M$ and which, together, satisfy (15). Then $\psi$ and $\lambda$ are necessarily exact 1-forms on $M$. The sets $P'$ and $Q'$ contain a certain amount of "triviality". For example, if $(g', \psi)$ satisfies (11) so also does $(\alpha g', \psi)$ $(0 \neq \alpha \in \mathbb{R})$ and if $(a, \lambda)$ satisfies (15) so also does $(\beta a, \beta \lambda)$ $(0 \neq \beta \in \mathbb{R})$. Thus if one defines a relation on each of the sets $P'$ and $Q'$ (each denoted by $\approx$ and readily shown to be an equivalence relation) by $(g_1', \lambda_1) \approx (g_2', \lambda_2) \Leftrightarrow g_2' = \alpha g_1'$ and $\lambda_1 = \lambda_2$, $(0 \neq \alpha \in \mathbb{R})$ and $(a_1, \lambda_1) \approx (a_2, \lambda_2) \Leftrightarrow a_2 = \beta a_1$ and $\lambda_2 = \beta\lambda_1$ $(0 \neq \beta \in \mathbb{R})$ then (13) and (14) give rise to a bijective correspondence between the quotient sets $P'/\approx$ and $Q'/\approx$. Each metric in an equivalence class in $Q'$ has the same connection.

Next, if $(a, \lambda) \in P'$ then so is $(a+h, \lambda)$ where $h$ is a type $(0,2)$ covariantly constant, symmetric tensor on $M$ satisfying the condition that $a + h$ is non-degenerate (and the choice $h = \gamma g$ for $\gamma$ in some open interval of $\mathbb{R}$ containing 0 satisfies this condition). If $(a_1, \lambda_1), (a_2, \lambda_2) \in P'$ write $(a_1, \lambda_1) \sim (a_2, \lambda_2) \Leftrightarrow a_2 = a_1 + h$ for some $h$ as above (equivalently, $\lambda_2 = \lambda_1$ from (15) since now $\nabla a_1 = \nabla a_2$ (see the remark after (15))). Then $\sim$ is also an equivalence relation on $P'$ and the equivalence class containing $(a, \lambda)$ is denoted by $[a, \lambda]$. The associated quotient set is $P'/\sim$.

Now let $P$ denote the set of all solutions to (15) where $a$ is restricted only by being a type $(0,2)$ symmetric tensor on $M$, no requirement of non-degeneracy being imposed. The equivalence relation $\sim$ described in the previous paragraph naturally extends to $P$ to give the quotient set $P/\sim$. Now the set $P$ has a natural structure of a real vector space according to the operations $(a, \lambda)+(b, \mu) = (a+b, \lambda+\mu)$ and $\gamma(a, \lambda) = (\gamma a, \gamma\lambda)$ $(\gamma \in \mathbb{R})$. The zero vector is $(0,0)$ and $-(a, \lambda) = (-a, -\lambda)$. The subset $T = \{(h, 0) \in P : \nabla h = 0\}$ of $P$ is a subspace (the *trivial subspace*) of $P$. It then follows that $P/\sim$ is the quotient space of $P$ with respect to $T$ and admits a real vector space structure according to the operations $[a, \lambda] + [b, \mu] = [a + b, \lambda + \mu]$ and $\gamma[a, \lambda] = [\gamma a, \gamma\lambda]$ which are easily checked to be well-defined and the zero of the vector space $P/\sim$ is

8

$[h, 0](= [g, 0] = [0, 0])$. Next, consider the map $\sigma : P \to P/\sim$ given by $\sigma : (a, \lambda) \to [a, \lambda]$. Clearly $\sigma$ is linear and its kernel, $\ker(\sigma)$, $(= T)$ is (isomorphic to) the vector space of all type $(0, 2)$, symmetric, covariantly constant tensors on $M$. Thus, $\dim \ker(\sigma) \geq 1$ and, since $\sigma$ is clearly onto, $\dim P = \dim(P/\sim) + \dim(\ker(\sigma))$. The necessity of dropping the non-degeneracy condition on $a$ to achieve the vector space structure on $P$ is clear. It will be seen later (theorem 2) that $P$ is finite dimensional.

Now apply the Ricci identity to $a$ and use (15) to get

$$(a_{ab;cd} - a_{ab;dc} =) a_{ae} R^e{}_{bcd} + a_{be} R^e{}_{acd} = g_{ac} \lambda_{bd} + g_{bc} \lambda_{ad} - g_{ad} \lambda_{bc} - g_{bd} \lambda_{ac} \tag{18}$$

where $\lambda_{ab} = \lambda_{a;b}(= \lambda_{ba})$. This leads to the following lemma which is (mostly) a special case of a more detailed result in [3, 23] and for which a definition is required. Suppose $m \in M$, that $F \in \Lambda_m M$ and that the curvature tensor $Riem$ of $(M, g)$ satisfies $R^{ab}{}_{cd} F^{cd} = \alpha F^{ab}$ ($\alpha \in \mathbb{R}$) at $m$, so that $F$ may be regarded as a (real) eigenvector of the map $f$ in (1) (recalling the liberties taken in identifying bivectors described in section 1). Then $F$ is called a (real) *eigenbivector of Riem at $m$ with eigenvalue $\alpha$*.

**Lemma 2.** *Let $(M, g)$ and $(M, g')$ be space-times with $g$ and $g'$ projectively related. Suppose at $m \in M$ that $F \in \Lambda_m M$ is a (real) eigenbivector of Riem of $(M, g)$ with zero eigenvalue (so that $F$ is in the kernel, $\ker f$, of $f$ in (1)). Then the blade of $F$ (if $F$ is simple) or each of the canonical pair of blades of $F$ (if $F$ is non-simple) is an eigenspace of the symmetric tensor $\nabla \lambda$ with respect to $g$ at $m$. (That is, if $p \wedge q$ is any of these blades ($p, q \in T_m M$) there exists $\mu \in \mathbb{R}$ such that for any $k \in p \wedge q$, $\lambda_{ab} k^b = \mu g_{ab} k^b$). In particular, if $\ker f$ is such that $T_m M$ is forced to be an eigenspace of $\nabla \lambda$ at each $m'$ in some connected open neighbourhood $U$ of $m$, then each of the following conditions is satisfied on $U$ for some $c \in \mathbb{R}$*

$$(a) \ \lambda_{ab} = c g_{ab}, \qquad (b) \ \lambda_d R^d{}_{abc} = 0, \qquad (c) \ a_{ae} R^e{}_{bcd} + a_{be} R^e{}_{acd} = 0 \tag{19}$$

*If (19) holds on $U$ either (i) $\lambda$ vanishes identically on $U$ or (ii) it does not, in which case any point of $U$ at which it does vanish is topologically isolated in $U$ (and in $M$) and so the subset of such points constitutes a closed subset of $U$ with empty interior. Further, Riem vanishes on some neighbourhood of any such point.*

**Proof**

The proof can mostly be found in [23] except for the last part. This follows by noting that, from (19)(a), if $c = 0$, $\lambda$ is covariantly constant on $U$ and either vanishes everywhere or nowhere on $U$. If $c \neq 0$, however, *the vector field with components $\lambda^a$ is proper homothetic with vanishing homothetic bivector on $U$*. The result now follows from [10] (theorem 10.6). $\square$

Lemma 2 gives important algebraic information about projective structure and it is remarked that any space-time $(M, g)$ which is either of any holonomy type except $R_9$, $R_{14}$ or $R_{15}$ or is of curvature class **B** **C** or **D** (at each $m \in M$) necessarily satisfies the conditions leading to (19) on $M$ (see table 1 for the relationship between the holonomy type and the curvature class). If such is the case, either $\lambda$ vanishes on $M$ ($\Leftrightarrow \nabla = \nabla'$) or $(M, g)$ admits a covariantly constant or proper homothetic vector field and a standard property of (any) homothetic vector field is that it either vanishes identically on $M$ or it cannot vanish on any non-empty open subset of $M$.

Next, some differential information will be provided which gives a uniqueness theorem for solutions of (15). Returning to (18) a contraction with $g^{ac}$ gives

$$4\lambda_{b;d} = \Psi g_{bd} + a^{ec} R_{ebcd} - a_{be} R^e{}_d \tag{20}$$

where $\Psi \equiv \lambda^a{}_{;a}$. Then a covariant differentiation of this last equation and use of (15) gives

$$4\lambda_{b;df} = g_{bd} \Psi_{,f} + a^{ec} R_{ebcd;f} + \lambda^e R_{ebfd} + \lambda^c R_{fbcd} - a_{be} R^e{}_{d;f} - \lambda_b R_{df} - g_{bf} \lambda_e R^e{}_d \tag{21}$$

where a comma denotes a partial derivative. The first term in (21) can, using the Ricci identity, be replaced by the terms $4(\lambda_{b;fd} + \lambda^e R_{ebdf})$. If this is done and the resulting equation contracted with $g^{bf}$ and use made of the contracted Bianchi identity for $(M, g)$ (which gives $R_{ebcd;f} g^{bf} = R_{ec;d} - R_{ed;c}$) one finds

$$3\Psi_{,d} = a^{ec}(R_{ec;d} - 2R_{ed;c}) - 10\lambda_e R^e{}_d \tag{22}$$

9

This equation, together with (20) and (15) give a first order system of differential equations for the quantities $\lambda_a, \Psi$ and $a_{ab}$. [This result has also been noticed in [6].] It is remarked here that the condition that $a$ be non-singular is not used in establishing (20) and (22) and so with (15), this system applies to the set $P$ of all solution pairs $(a, \lambda)$ of (15). This leads to the following theorem.

**Theorem 2.**

(i) If $(a, \lambda), (b, \mu) \in P$ and there exists $m \in M$ such that $a(m) = b(m)$, $\lambda(m) = \mu(m)$ and $\Psi(m) = \Psi'(m)$ (where $\Psi' = \mu^a_{;a}$) then $a = b$ and $\lambda = \mu$ on $M$ (that is, $(a, \lambda) = (b, \mu)$ on $M$).

(ii) The vector space $P$ is finite dimensional and $\dim P \leq 15$.

(iii) If $(a, \lambda), (b, \mu) \in P$ and if there exists a non-empty open subset $U \subset M$ such that $b = a + \alpha g$ $(\alpha \in \mathbf{R})$ on $U$ then $b = a + \alpha g$ and $\lambda = \mu$ on $M$. In particular, if $a = b$ on $U$ then $a = b$ and $\lambda = \mu$ on $M$ and so $(a, \lambda) = (b, \mu)$.

**Proof**

(i) Let $I$ be an open interval of $\mathbf{R}$ and $c : I \to M$ a smooth curve in $M$ passing through $m$. Let $m' \in c(I)$ and choose a chart with coordinate functions $x^a$ about $m'$. If $s$ is a parameter for $c$ and $k^a = d/ds(x^a \circ c)$ is the tangent to $c$ within the chart domain, a contraction of (15), (20) and (22) with $k^c$ leads to first order differential equations for the quantities $\lambda_a$, $\Psi$ and $a_{ab}$. Thus if the conditions of (i) hold at $m'$ the theory of such equations show that they will hold in an open neighbourhood in $c(I)$ of $m'$. On the other hand, if the conditions of (i) fail at $m'$ they will, by continuity, fail on some open neighbourhood in $c(I)$ of $m'$. Since $c(I)$ is a connected subset of $M$ and since the conditions (i) are given to hold at $m$, they must hold on $c(I)$. Since $M$ is a manifold whose topology is connected, it is also path connected and so any point in $M$ may be joined to $m$ by such a curve. The result now follows.

(ii) This is immediate since the number of independent choices of $\lambda_a(m), \Psi(m)$ and $a_{ab}(m)$ is $4 + 1 + 10 = 15$.

(iii) If the conditions of (iii) hold it is clear from (15) that $\lambda$ and $\mu$ agree on $U$. Hence the conditions of part (i) above hold at each point of $U$ for $(a + \alpha g, \lambda)$ and $(b, \mu)$ and the result follows. $\square$

It is clear that this theorem is true (with (ii) obviously modified) for $(M, g)$ of arbitrary signature and dimension.

Two remarks may be made regarding these results. First, if $(M, g)$ is an Einstein space, the above differential systems reduce to [5]

$$\lambda_{ab;c} = 2g_{ab}r_c + g_{ac}r_b + g_{bc}r_a \qquad \left(r_a = -\frac{R}{12}\lambda_a\right) \qquad (23)$$

and which is precisely the condition that $\lambda$ is a *projective* (co)vector field on $(M, g)$ (and is proper projective if $\lambda$ is not identically zero on $M$) and an *affine* (co)vector field in the vacuum (Ricci flat) case when the constant $R = 0$. Thus $\lambda$ vanishes on $M$ if it vanishes on any non-empty open subset of $M$ (see, e.g. [10]). Thus, for Einstein spaces, the first order system has essentially decoupled since (23) no longer involves $a$. In this case one must try to find a solution of (23) for $\lambda$ and where, in addition, $\lambda$ is an exact 1-form on $M$ and then, if such a $\lambda$ can be found, to find a corresponding solution of (15) for $a$. The details of such spaces are known [5, 8]. [It is remarked here that if $(M, g)$ is a space-time which is an Einstein space whose Weyl tensor does not vanish over any non-empty open subset of $M$ then it cannot admit a proper (that is, not affine) projective vector field [22]] Second, if the conditions of lemma 2, including (19), hold then decoupling again takes place and the above first order system reduces to the homothetic condition for $\lambda$ on $U$ with zero homothetic bivector. Then $\lambda$ is a homothetic (co)vector field on $M$ and vanishes on $M$ if it vanishes on any non-empty open subset of $M$. [The decoupling referred to here is perhaps more obvious if one writes the first order system above in terms of (the 24 independent components of) $\lambda$, $\nabla\lambda$ and $a$ and which can be achieved in a similar manner to that for the original first order system.] Thus, if $(M, g)$ is an Einstein space or satisfies (19) in lemma 2 it can only admit a non-trivially projectively related partner $(M, g')$ if it admits some kind of non-trivial symmetry vector field.

10

## 5. Projective Structure and holonomy; general results

In this section a summary of some known results and proofs of some new ones is given and which relate the holonomy type of a space-time $(M, g)$ to the collection of space-times $(M, g')$ where $g'$ is (locally or globally) projectively related to $g$. Since the holonomy type of the original space-time may not be known, these results will also be given in terms of the curvature class and also in terms of the rank of the map $f$ (that is, the dimension of the spaces $B_m$) in (1). This can be done quite efficiently using the holonomy theory described in section 3 and allows for easier access to them directly through the curvature tensor. First the following technical lemma is required and which was given in [23].

**Lemma 3.**

(i) Let $X$ be a topological space and let $A$ and $B$ be disjoint subsets of $X$ such that $A$ and $A \cup B$ are open in $X$ and $A \cup B$ is dense in $X$. Suppose $B = B_1 \cup B_2$, with $B_1$ and $B_2$ disjoint, and $\text{int} B \subset \text{int} B_1 \cup B_2$ where $\text{int}$ denotes the interior operator in $X$. Then $X$ may be disjointly decomposed as $X = A \cup \text{int} B_1 \cup \text{int} B_2 \cup J$ where $J$ is the closed subset of $X$ defined by the disjointness of the decomposition and $A \cup \text{int} B_1 \cup \text{int} B_2$ is open and dense in $X$ (that is, $\text{int} J = \emptyset$).

(ii) Let $X$ be topological space and let $A_1, ..., A_n$ be disjoint subsets of $X$ such that, for $k = 1, ...n$, $\cup_{i=1}^{i=k} A_i$ are open subsets of $X$ and such that $\cup_{i=1}^{i=n} A_i$ is (open and) dense in $X$. Then $X$ may be disjointly decomposed as $X = A_1 \cup \text{int} A_2 \cup ... \cup \text{int} A_n \cup K$ where $K$ is the closed subset of $X$ defined by the disjointness of the decomposition and $A_1 \cup \text{int} A_2 \cup ... \cup \text{int} A_n$ is open and dense in $X$ (that is, $\text{int} K = \emptyset$).

**Theorem 3.** Let $(M, g)$ and $(M, g')$ be space-times with $\nabla$ and $\nabla'$ projectively related.

(i) Suppose $(M, g)$ is of holonomy type $R_2$, $R_3$, $R_4$, $R_6$, $R_7$, $R_8$ or $R_{12}$. Then $\nabla = \nabla'$ on $M$.

(ii) Suppose $(M, g)$ has holonomy type $R_{10}$, $R_{11}$ or $R_{13}$ and that there exists $m \in M$ at which the curvature rank is 2 or 3 (equivalently, at which the curvature class is **C**). Then $\nabla = \nabla'$ on $M$.

(iii) If $M$ admits a non-empty, connected open subset $U$ such that the only solution of (15) on $U$ is $\lambda = 0$ and $a = \alpha g$ ($0 \neq \alpha \in \mathbf{R}$) then the only solution of (15) on $M$ is $\lambda = 0$ and $a = \alpha g$ and so $\nabla = \nabla'$. In particular, if $U$ may be chosen so that the holonomy type of $(U, g)$ is either $R_9$ and the curvature rank is 1 or 2 at each point of an open dense subset of $U$ (equivalently, Riem is of curvature class **C** or **D** at each point of this open dense subset), or, $R_{14}$ and the curvature class is **B** or **C** at each point of an open dense subset of $U$, $\nabla = \nabla'$ on $M$. Also, if $(M, g)$ is of holonomy type $R_9$ and Riem has curvature rank 1 or 2 at each point of an open dense subset $U$ of $M$ (equivalently, Riem is of curvature class **C** or **D** at each point of $U$), $\nabla = \nabla'$ on $M$ and if $(M, g)$ is of holonomy type $R_{14}$ and Riem has curvature class **B** or **C** at each point of an open dense subset $U$ of $M$, $\nabla = \nabla'$ on $M$.

**Proof**

(i) The holonomy conditions imposed show that $(M, g)$ is not flat and so there exists $m \in M$, and hence an open, connected neighbourhood $U$ of $m$, such that Riem is nowhere zero on $U$. It also follows from the remarks after lemma 2 that the conditions of this lemma leading to (19) hold at each point of $M$ and so $\lambda$ is a homothetic (co)vector field on $M$. Now the holonomy algebra of $(U, g)$ is a subalgebra of that of $(M, g)$ and it follows from table 1 that for each holonomy algebra listed in part (i), its subalgebras are also contained in this list (excluding, of course, the $R_5$ subalgebra of $R_{12}$ which cannot occur as the holonomy algebra of a space-time). Thus one may proceed with a hierarchical proof, using theorems 2, 3 and 5 in [23] applied to the restricted space-time $(U, g)$. It follows from this that, in each case, $\lambda = 0$ on $U$. Since $\lambda$ is homothetic and vanishes on $U$ it vanishes on $M$ and so $\psi = 0$ (and $\nabla = \nabla'$) on $M$.

(ii) If there exists $m \in M$ where the curvature rank is 2 or 3 (and since the maximum curvature rank for these holonomy types is 3) then by one of the many versions of the rank theorem there exists a connected, open neighbourhood $U$ of $m$ such that the curvature rank is 2 or 3 (and hence the curvature class is **C**) at each point of $U$. One now proceeds as in part (i) to see that (19) holds on $M$ and so, applying theorem 4 of [23] to $(U, g)$, $\lambda$ vanishes on $U$ and hence on $M$. Thus $\nabla = \nabla'$ on $M$.

(iii) The first part of (iii) is clear from theorem 2(iii) since the pair $(\alpha g, 0)$ is a global solution of (15) on $M$ and agrees with any other such solution on $U$.

Next suppose that $(U, g)$ is of holonomy type $R_9$. Then it is noted from table 1 that if $Riem$ is of curvature class $\mathbf{C}$ at $m \in U$, it is necessarily of curvature rank 2 at $m$ with the vector $r$ in the definition of this curvature class either null or spacelike. Conversely, if $Riem$ is of curvature rank 2 at $m$ it is necessarily of curvature class $\mathbf{C}$ since all members of $B_m$ are, for this holonomy type, simple with the null vector $l$ in their blade. It is also remarked that if the curvature class is $\mathbf{D}$ at any $m \in M$, (3) holds with $l$ in the blade of $F$. So let $J$, $C_{nn}$, $C_n$, $D_{nn}$ and $D_n$ denote, respectively, the subsets of points of $U$ at which $Riem$ vanishes, is of (curvature) class $\mathbf{C}$ with $r$ spacelike, is of class $\mathbf{C}$ with $r$ null, is of class $\mathbf{D}$ with $F$ timelike in (3) and is of class $\mathbf{D}$ with $F$ null in (3). Define the sets $C \equiv C_{nn} \cup C_n$ and $D \equiv D_{nn} \cup D_n$ so that, by the rank theorem, $C$ and $C \cup D$ are open in $U$ and, in addition, $C \cup D$ is dense in $U$. Then consider the disjoint decomposition

$$U = C \cup \mathrm{int} D_{nn} \cup \mathrm{int} D_n \cup K \tag{24}$$

where the interior operator is here taken in the subspace topology of $U$ (but, since $U$ is open in $M$, this is the same as the interior operator in $M$). In this decomposition the closed subset $K$ (in the topology of $U$) is determined by the disjointness of the decomposition. Also, $C_{nn}$ is open in $C$ (and hence in $M$) because if $C_{nn} \neq \emptyset$ and $m \in C_{nn}$, $B_m$ contains a simple timelike member, say $F$. So let $G$ be a smooth bivector field defined on some open neighbourhood of $m$ and such that $f(G)(m) = F$. Then $f(G)$ is timelike over some neighbourhood $V$ of $m$. Since $C$ is open in $U$ one may choose $V \subset C$ and so $V \subset C_{nn}$. It follows that $C_{nn}$ is open in $C$ and hence in $U$. If $\mathrm{int} D \neq \emptyset$ then for $m \in \mathrm{int} D$ there is a connected open neighbourhood $V'$ of $m$ contained in $\mathrm{int} D$ on which $Riem$ may be written as in (3) with $F$ smooth (see [10]). Then a consideration of the smooth function $F^{ab} F_{ab}$ on $V'$ reveals that $F^{ab} F_{ab}(m) < 0$ for $m \in D_{nn}$ and $F^{ab} F_{ab}(m) = 0$ for $m \in D_n$. Thus $m \in \mathrm{int} D \Rightarrow m \in \mathrm{int} D_{nn}$ or $m \in D_n$ and so $\mathrm{int} D \subset \mathrm{int} D_{nn} \cup D_n$. Since $C$ and $C \cup D$ are each open in $U$ and $C \cup D$ is dense in $U$, lemma 3 (i) shows that the closed (in $U$) set $K$ has empty interior (in $U$) and hence that $U \setminus K$ is open and dense in $U$. It then suffices to show for the first part of the proof that under the relevant conditions of part (iii), $\lambda$ vanishes on each of $C_{nn}$, $\mathrm{int} C_n$, $\mathrm{int} D_{nn}$ and $\mathrm{int} D_n$ (because $\lambda$ then vanishes on $C_n$ and hence on $C$) to establish that $\lambda$ vanishes on $U \setminus K$ and hence on $U$.

So suppose $C_{nn} \neq \emptyset$ and let $m \in C_{nn}$. Then, by the $R_9$ holonomy type assumption, it can be checked that there exists an open neighbourhood $V \subset C_{nn}$ of $m$ and smooth vector fields $l, n, x$ and $y$ on $V$ such that $l$ is null and recurrent on $V$ (say $\nabla l = l \otimes p$ for some 1-form field $p$ on $V$), such that $l, n, x$ and $y$ constitute a null tetrad at each point of $V$ and such that $l \wedge n$ and $l \wedge x$ span $B_{m'}$ at each $m' \in M$ (and so $R^a{}_{bcd} y^d = 0$ on $V$). Then, from (1), $l \wedge x$, $l \wedge y$, $x \wedge y$ and $n \wedge y$ span $\ker(f_{m'})$ at each $m' \in V$ and the conditions and conclusions of lemma 2 hold. Thus $\lambda_{a;b} = c g_{ab}$ for some constant $c$ and $a$ satisfies (19(c)). So theorem 1(ii) gives on $V$

$$a_{ab} = \phi g_{ab} + \mu y_a y_b \tag{25}$$

where $\phi$ and $\mu$ are functions whose smoothness follows from that of $a_{ab} y^a y^b$ and $g^{ab} a_{ab}$. A substitution into (15) (recalling that a comma denotes a partial derivative) then reveals

$$\phi_{,c} g_{ab} + \mu_{,c} y_a y_b + \mu(y_{a;c} y_b + y_a y_{b;c}) = g_{ac} \lambda_b + g_{bc} \lambda_a \tag{26}$$

Contractions of (26) with $l^a x^b$, $l^a y^b$ and $n^a x^b$ easily show that $\lambda_a l^a = \lambda_a n^a = \lambda_a x^a = \lambda_a y^a = 0$ on $V$ and so $\lambda = 0$ on $V$. It follows that $\lambda = 0$ on $C_{nn}$.

If $\mathrm{int} C_n \neq \emptyset$ let $m \in \mathrm{int} C_n$ and choose an open neighbourhood $V \subset C_n$ of $m$ and smooth vector fields $l, n, x$ and $y$ on $V$ as before such that $l$ is recurrent, $l \wedge x$ and $l \wedge y$ span $B_{m'}$ for each $m' \in V$ and $R^a{}_{bcd} l^d = 0$ on $V$. Then $l \wedge x$, $l \wedge y$, $l \wedge n$ and $x \wedge y$ span $\ker(f_{m'})$ at each $m' \in V$ and lemma 2 again applies. Thus theorem 1(ii) shows that

$$a_{ab} = \phi g_{ab} + \mu l_a l_b \tag{27}$$

for functions $\phi$ and $\mu$ on $V$ which are smooth since $a_{ab} n^a n^b$ and $g^{ab} a_{ab}$ are. Sinjukov's equation (15) then gives

$$\phi_{,c} g_{ab} + 2\mu l_a l_b p_c + \mu_{,c} l_a l_b = g_{ac} \lambda_b + g_{bc} \lambda_a \tag{28}$$

12

and contractions with $l^a x^b$ and $n^a y^b$ again show that $\lambda = 0$ on $V$ and hence on $\mathrm{int}C_n$. [There is an alternative proof here that will prove useful in other cases to follow. The condition $R^a{}_{bcd}l^d = 0$ is easily seen to hold for this case and the recurrence condition and Ricci identity for $l$ show that $l_{a;[bc]} = 0$ and hence that $p_{[a;b]} = 0$ on $V$. Thus the 1-form $p$ is closed and hence is locally a gradient on $V$. Thus, by reducing $V$ if necessary, one may write $p_a = \alpha_{,a}$ for some function $\alpha$ on $V$ and then $e^{-\alpha}l^a$ is covariantly constant on $V$. Then the proof for the holonomy type $R_8$ given in [23] shows that $\lambda = 0$ on $V$ and hence on $\mathrm{int}C_n$.]

If $\mathrm{int}D_{nn} \neq \emptyset$ let $m \in \mathrm{int}D_{nn}$ and choose an open neighbourhood $V \subset D_{nn}$ of $m$ and smooth vector fields $l, n, x$ and $y$ on $V$ as before with $l$ recurrent and with $l \wedge n$ spanning $B_{m'}$ at each $m' \in V$ and $R^a{}_{bcd}x^d = R^a{}_{bcd}y^d = 0$ on $V$. Again lemma 2 applies and theorem 1(i) gives

$$a_{ab} = \phi g_{ab} + \mu x_a x_b + \nu y_a y_b + \rho(x_a y_b + y_a x_b) \tag{29}$$

for functions $\phi$, $\mu$, $\nu$ and $\rho$ which are then seen to be smooth since $a_{ab}x^a x^b$, $a_{ab}y^a y^b$, $a_{ab}x^a y^b$ and $a_{ab}g^{ab}$ are. Sinjukov's equation (15) then reveals that

$$\phi_{,c}g_{ab} + \mu_{,c}x_a x_b + \mu(x_{a;c}x_b + x_a x_{b;c}) + \nu_{,c}y_a y_b + \nu(y_{a;c}y_b + y_a y_{b;c})$$
$$+\rho_{,c}(x_a y_b + y_a x_b) + \rho(x_{a;c}y_b + x_a y_{b;c} + y_{a;c}x_b + y_a x_{b;c}) = g_{ac}\lambda_b + g_{bc}\lambda_a \tag{30}$$

Noting that, from the recurrence condition on $l$, $l^a x_{a;b} = l^a y_{a;b} = 0$ on $V$, a contraction of (30) with $l^a$ and a simple rank argument gives $\lambda_a l^a = 0$ and $\phi_{,a}l_b = l_a \lambda_b$, whilst a contraction with $n^a n^b$ gives $\lambda_a n^a = 0$. Hence $\lambda = 0$ on $V$ and hence on $\mathrm{int}D_{nn}$.

If $\mathrm{int}D_n \neq \emptyset$ let $m \in \mathrm{int}D_n$ and choose an open neighbourhood $V \subset D_n$ of $m$ and smooth vector fields $l, n, x$ and $y$ on $V$, as before, such that $l$ is recurrent and such that $l \wedge x$ spans $B_{m'}$ at each $m' \in \mathrm{int}D_n$ (and note that $R^a{}_{bcd}l^d = R^a{}_{bcd}y^d = 0$ on $V$). The same procedure as in the previous cases together with contractions of the final equation with $l^a$ and $x^a x^b$ lead to $\lambda_a l^a = 0$, $\phi_{,a}l_b = l_a \lambda_b$ and $\phi_{,a} = 2(\lambda_b x^b)x_a$ from which it follows that $\lambda = 0$ on $V$ and hence on $\mathrm{int}D_n$.

It follows that $\lambda = 0$ on $U$ and hence, from (15) that $\nabla a = 0$ on $U$. But since $(U, g)$ is of holonomy type $R_9$, it follows from remarks in section 3 that the only solution of $\nabla a = 0$ on $U$ is $a = \alpha g$ for $\alpha \in \mathbb{R}$. The result now follows from the first sentence of part (iii) of this theorem.

If $(U, g)$ has holonomy type $R_{14}$ the restriction that $Riem$ must be of curvature class $\mathbf{B}$ or $\mathbf{C}$ over an open dense subset $U'$ of $U$ means that the curvature rank is either 2 or 3 over $U'$. The possible spanning bivectors in $B_m$ for $m \in U'$ (table 1) show that, in the notation of table 1, the simple members of $B_m$ cannot contain both $l \wedge n$ and $x \wedge y$ whilst the null members cannot contain either $l \wedge n$ or $x \wedge y$. [To see this, use the fact that a non-zero bivector $H \in \Lambda_m M$ is simple $\Leftrightarrow H_{ab}\overset{*}{H}{}^{ab} = 0$ and is null $\Leftrightarrow H_{ab}H^{ab} = H_{ab}\overset{*}{H}{}^{ab} = 0$.] Thus $U$ may be disjointly decomposed as follows

$$U = C^3 \cup B \cup C_{nn}^2 \cup C_n^2 \cup J \tag{31}$$

where $C^3$, $B$, $C_{nn}^2$, $C_n^2$ and $J$ are subsets of $U$ with $C^3$ the subset of points $m'$ at which the curvature class is $\mathbf{C}$, $\dim B_{m'} = 3$ and $r$ in the definition of curvature class $\mathbf{C}$ in section 2 is (necessarily) null, $B$ the subset of points $m' \in U$ at which the curvature class is $\mathbf{B}$, $\dim B_{m'} = 2$, $C_{nn}^2$ the subset of points $m' \in U$ at which the curvature class is $\mathbf{C}$, $\dim B_{m'} = 2$ and $r$ spacelike, $C_n^2$ the subset of points $m' \in M$ at which the curvature class is $\mathbf{C}$, $\dim B_{m'} = 2$ and $r$ is null and $J$ is closed with empty interior. (Again all topological statements are made in the subspace topology of $U$.) The set $C^3$ is open in $U$ by the rank theorem as is $C^3 \cup B$ (since $B_{m'}$ contains non-simple members if and only if $m' \in B$ and so, for $m' \in B$ and for some smooth bivector $H$ defined on some open neighbourhood of $m'$, $f(H)(m')$ is a non-simple member of $B_{m'}$ and $f(H)$ is non-simple over some open neighbourhood of $m'$). Clearly, if $C^2 \equiv C_{nn}^2 \cup C_n^2$, then $C^3 \cup B \cup C^2$ is open in $U$ and use of the function $F^{ab}F_{ab}$ (as in an argument above) or, if $\mathrm{int}C^2 \neq \emptyset$, of the possibility of choosing a smooth (vector field) solution $k$ of the equation $R^a{}_{bcd}k^d = 0$ on $\mathrm{int}C^2$, shows that $\mathrm{int}C^2 \subset \mathrm{int}C_{nn}^2 \cup C_n^2$. Thus one has the disjoint decomposition given by

$$U = C^3 \cup B \cup \mathrm{int}C_{nn}^2 \cup \mathrm{int}C_n^2 \cup K \tag{32}$$

13

in which $K$ is closed and has empty interior in $U$, from lemma 3(i). Thus, to show $\lambda$ vanishes on $U$, it is sufficient to show that $\lambda$ vanishes on each interior in (32) (including $C^3 \cup B$) and hence on the open dense subset $U \setminus K$ of $U$.

If $C^3 \neq \emptyset$, let $m \in C^3$ and choose an open neighbourhood $V \subset C^3$ of $m$ on which vector fields $l, n, x$ and $y$ are defined, as before, with $l$ recurrent. It can then be checked that these vector fields may be chosen such that for each $m' \in C^3$, $B_{m'}$ is spanned by $l \wedge x$ $l \wedge y$ and $x \wedge y$, evaluated at $m'$. Thus, $R^a{}_{bcd}l^d = 0$ on $V$ and the Ricci identity may be used, as in the (alternative) proof following (28) above, to show that $V$ may be reduced, if necessary, so that $l$ is covariantly constant on $V$. It then follows from the proof for the holonomy type $R_{11}$ in [23] that $\lambda = 0$ on $V$ and hence on $C^3$.

If $B \neq \emptyset$ the proof that $\lambda = 0$ on $B$ follows from the proof in the holonomy type $R_7$ case in [23] (theorem 3). [It is remarked here that, in [23], it was stated that a space-time of curvature class **B** had holonomy type $R_7$. This is true if $\alpha$, $\beta$ and $F$ are smooth in (2) but is not proven if they are not. However, theorem 3 in [23] still holds since $\alpha$, $\beta$ and $F$ can be shown to be smooth over an open dense subset of $M$ on which $\lambda$ then vanishes and so $\lambda = 0$ on $M$. In the present case, $(M, g)$ has holonomy type $R_{14}$ and the local existence of the smooth recurrent vector field $l$ and hence of the associated smooth local null tetrad reveals the smoothness of $\alpha$, $\beta$ and $F$ on $B$ and the result that $\lambda = 0$ on $B$ follows.] If $\mathrm{int}C_{nn}^2 \neq \emptyset$ the proof that $\lambda = 0$ on $\mathrm{int}C_{nn}^2$ is the same as that given above for holonomy type $R_9$ on the subset $C_{nn}$. If $\mathrm{int}C_n^2 \neq \emptyset$, the proof that $\lambda = 0$ on $\mathrm{int}C_n^2$ is as for the holonomy type $R_{11}$ (curvature class **C**) given in [23]. It follows that $\lambda = 0$ on $U$ and, since $(U, g)$ is of holonomy type $R_{14}$, an argument similar to that in the $R_9$ case shows that the only solutions of (15) on $U$ are of the form $\lambda = 0$ and $a = \alpha g$ for $\alpha \in \mathbb{R}$. The result now follows.

The proof of the two statements in the final sentence of part (iii) now follows from the previous results. $\square$

The last theorem can be restated in many other ways involving, say, curvature class or curvature rank instead of holonomy type. To avoid too much repetition, the following theorems single out some special cases.

**Theorem 4.** *Let $(M, g)$ and $(M, g')$ be space-times with $\nabla'$ and $\nabla$ projectively related.*

(i) *Suppose $(M, g)$ is of curvature class **D** (equivalently curvature rank 1) on some open, dense subset $U$ of $M$ and of holonomy type $R_2, R_3, R_4, R_6, R_7, R_8, R_9$ or $R_{12}$. Then $\nabla' = \nabla$ on $M$.*

(ii) *Suppose $(M, g)$ is of curvature class **C** on some open, dense subset $U$ of $M$ and of any permissible holonomy type except $R_{15}$. Then $\nabla' = \nabla$ on $M$.*

(iii) *Suppose $(M, g)$ is of curvature class **B** on some open, dense subset $U$ of $M$. Then $\nabla' = \nabla$ on $M$.*

(iv) *Suppose that $(M, g)$ is such that at each point $m$ of some non-empty, open, subset $U$ of $M$, the curvature class is **A** and $(\ker f)_m$ is such that (19) holds at $m$. Then $\nabla' = \nabla$ on $M$.*

**Proof**

(i) For the holonomy types $R_2, R_3, R_4, R_6, R_7, R_8$ and $R_{12}$ the result follows trivially from theorem 3(i). For the type $R_9$ it follows from theorem 3(iii).

(ii) The curvature class **C** condition and the consequent fact that the curvature rank is $\geq 2$ at each $n \in U$ means that the holonomy type is not $R_7$ and, of course, cannot be $R_2, R_3$ or $R_4$. For types $R_6, R_8$, $R_{10}$ $R_{11}, R_{12}$ and $R_{13}$, the result follows trivially from theorems 3(i) and 3(ii) and for types $R_9$ and $R_{14}$ it follows from theorem 3(iii).

(iii) This follows by using lemma 2 and (19) to show that $\lambda = 0$ on $U$ and hence on $M$.

(iv) This is clear since the curvature class **A** restriction on $U$ means that the only solution to (19c) is when $a$ is proportional to $g$ on $U$ [10, 15]. On substituting this into (15) and performing some obvious contractions one finds that $\lambda = 0$ on $U$ and the result now follows from the first part of theorem 3(iii).

**Theorem 5.** *Let $(M, g)$ and $(M, g')$ be space-times with $\nabla'$ and $\nabla$ projectively related.*

(i) *Suppose Riem has curvature rank 2 at each point of some open dense subset $U$ of $M$ and $(M, g)$ has any permissible holonomy type except $R_{15}$. Then $\nabla' = \nabla$ on $M$.*

14

*(ii) Suppose Riem has curvature rank 3 at each point of some open dense subset $U$ of $M$, that $(\ker f)_m$ is such that lemma 2 holds and (19) is satisfied at each $m \in U$ and that $(M, g)$ has any permissible holonomy type except $R_{15}$. Then $\nabla' = \nabla$ on $M$.*

**Proof**

(i) Since *Riem* has curvature rank 2 at each point of $U$ one may decompose $M$, disjointly, as $M = A \cup B \cup C \cup J$ where $A$ (respectively, $B$, $C$ and $J$) denote the subsets of $M$ at each point of which the curvature class is **A** (respectively **B**, **C** and **O**) and int$J = \emptyset$. Then $M$ admits the disjoint decomposition

$$M = A \cup \mathrm{int}B \cup \mathrm{int}C \cup K \tag{33}$$

where $A$ is open in $M$. Since $A \cup B$ and $A \cup B \cup C$ are open in $M$, with the latter dense in $M$, it follows from lemma 3(ii) that the closed set $K$ has empty interior in $M$ and hence that $M \setminus K$ is open and dense in $M$. Since $\dim \ker f = 4$ at each $m \in U$ it can be shown that $B_m$ always contains a non-simple member ([10], p 392) and the conclusions of lemma 2 can be checked to apply on $U$. Then theorem 4 parts (iv), (iii) and (ii) reveal that $\lambda$ vanishes on $A$, int$B$ and int$C$, respectively, if any of these is non-empty. Thus $\lambda = 0$ on the open dense subset $M \setminus K$ of $M$ and hence on $M$.

(ii) Under the conditions stipulated here the curvature class is either $A$ or $C$ at each point of $U$. One then decomposes M disjointly in an obvious notation as $M = A \cup C \cup J = A \cup \mathrm{int}C \cup K$ where $J$ is closed and has empty interior in $M$ and, since $A$ and $A \cup C$ are open in $M$ with the latter dense in $M$, lemma 3(ii) shows that the closed set $K$ has empty interior in $M$. So $M \setminus K$ is open and dense in $M$. The result now follows from theorem 4 part (ii) (for int$C$) and part (iv) for $A$. $\square$

In the event that the projective related condition linking $(M, g')$ and $(M, g)$ leads to $\nabla' = \nabla$ an argument from holonomy theory can then be used to find the relationship between $g'$ and $g$ [10, 23].

An inspection of theorems 3, 4 and 5 reveals certain gaps in them which will now be identified properly. As before, let $(M, g)$ and $(M, g')$ be space-times, with $\nabla$ and $\nabla'$ projectively related. First consider the situation when $(M, g)$ has holonomy type $R_{10}$, $R_{11}$ or $R_{13}$. It is clear from theorem 3(ii) that a full resolution of these holonomy types requires only a consideration of the situation when the curvature rank is $\leq 1$ at each point of $M$, that is, when $M$ is of curvature class **D** or **O** at each of its points. This was completed in [23] where it was shown that, locally or globally, metrics $g$ and $g'$ exist which are projectively related but where $\nabla \neq \nabla'$ and, given $g$, all such metrics $g'$ may be found. The proof in [23] can be improved a little by noting from the last part of lemma 2 that, if $U$ is the non-empty open subset of $M$ on which *Riem* does not vanish, whether $c$ is zero or not in (19), $\lambda$ is either identically zero on $M$ or nowhere zero on $U$. The problem of local projective relatedness on $M$ can then be resolved on some neighbourhood of any $m \in U$. If $M$ is non-flat one may take $U$ open and dense in $M$. This completes the situation for holonomy types $R_{10}$, $R_{11}$ and $R_{13}$.

For holonomy type $R_9$, and with the non-flat condition assumed on $(M, g)$ for simplicity, one sees that, at each point of an open dense subset $U$ of $M$, the curvature class of *Riem* may be **A**, **C** or **D** and theorem 3(iii) shows that $\nabla = \nabla'$ on $M$ except possibly when there exists $m \in M$ at which the curvature class is **A**. In this case, since the subset of points of $M$ at which the curvature class is **A** is open in $M$ ([10], page 393) one is led to consider the situation when $M$ admits a non-empty open subset on which the curvature class is **A**. This will be considered in the next section.

For holonomy type $R_{14}$ and with the non-flat condition again assumed on $(M, g)$ one sees that, at each point of an open dense subset $U$ of $M$, the curvature class may be any of the classes **A**, **B**, **C** or **D**. Theorem 3(iii) shows that $\nabla = \nabla'$ on $M$ except possibly when the curvature class is **A** at some $m \in U$ and hence in some open neighbourhood of $m$ contained in $U$, or the curvature class is **D** at each point of some non-empty open subset $V$ of $U$. The first of these cases will be dealt with in the next section. For the second possibility, (3) holds on $V$ so let $V'$ be any (necessarily open) component of $V$ and consider the space-time $(V', g)$, decomposing it, in a manner done several times above, into regions where $F$ in (3) is spacelike, timelike or null. The techniques used for holonomy type $R_6$ in [23] or $R_9$ above show that $\lambda = 0$ on $V$ except, possibly, when $F$ is spacelike over some non-empty open subset of $V$. This latter case then becomes essentially the same as a subcase of the holonomy type $R_{11}$ and is dealt with in [23] as mentioned above.

For curvature class **C** theorem 4(ii) will be completed in the next section by exhibiting an example of a space-time $(M, g)$ of curvature class **C** and holonomy type $R_{15}$ which admits a projectively related metric $g'$ for which $\nabla \neq \nabla'$. [The well-known set of FRWL cosmological metrics gives examples of projectively related, but distinct, Levi-Civita connections of curvature class **A** and holonomy type $R_{15}$ [4]]

## 6. Projective Structure and Holonomy; Special Cases

The only space-time holonomy group possibilities that have not been "essentially" completely described (that is, neglecting in some cases the possibility of non-empty flat regions) by the previous theorems are the types $R_9$, $R_{14}$ and $R_{15}$. These three holonomy types will now be discussed and shown to reveal projective relationships between Levi-Civita connections which do *not* result in the equality of these connections.

*6.1. Space-times admitting a normal conformal vector field*

It is convenient, at this point, to introduce a lemma which is a special case of the following result [24, 10]. If $M$ is a manifold of arbitrary signature $n \geq 2$, $g$ is a metric on $M$ of arbitrary signature and $M$ admits a global, nowhere zero or null, hypersurface-orthogonal conformal vector field $X$ then $X$ is covariantly constant with respect to the Levi-Civita connection of the conformally related global metric $\tilde{g} = |g(X, X)|^{-1}g$ on $M$. A similar *local* result holds if $X$ is everywhere null. [Here, hypersurface-orthogonal means that the covector field associated with $X$ is locally proportional to a gradient but with the proportionalty factor not necessarily constant.]

**Lemma 4.** *Let $(M, g)$ be a space-time and $X$ a global, nowhere zero nor null, smooth vector field on $M$ (and with associated covector field denoted by $\tilde{X}$) satisfying $\nabla \tilde{X} = \sigma g$ for a smooth function $\sigma$ on $M$.*

*(i) For any $m \in M$ there exists a coordinate neighbourhood $U$ of $m$ with coordinates functions $u, x^\alpha$ ($\alpha, \beta = 1, 2, 3$) such that $g$ is given on $U$ by*

$$ds^2 = \epsilon du^2 + \rho^2 h_{\alpha\beta} dx^\alpha dx^\beta \tag{34}$$

*where $\epsilon = \pm 1$, $h$ is a smooth metric on the level surfaces of $u$ in $U$ and of signature $(+, +, +)$ for $\epsilon = -1$ and $(-, +, +)$ for $\epsilon = 1$, where $\rho$ and $\sigma$ are smooth functions on $U$ depending only on $u$ and $\rho^2 = |g(X, X)|$, $\sigma = d\rho/du$ and $X = \rho\partial/\partial u$.*

*(ii) If $\sigma$ is a non-zero constant function on $M$ (so that $X$ is homothetic on $M$) then, after a rescaling of $X$ and a translation of the coordinate $u$, one may take $\sigma = 1$ and (34) holds with $\rho = u$.*

**Proof**

The conditions of the theorem show that $X$ is a nowhere zero nor null conformal vector field (since $\mathcal{L}_X g = 2\sigma g$) with zero conformal bivector and that $\tilde{X}$ is a locally a gradient. Thus the remarks preceding lemma 4 show that $X$ is covariantly constant with respect to the Levi-Civita connection, $\nabla'$, of $\tilde{g} = \rho^{-2}g$; $\nabla' X = 0$. By choosing local coordinates $t, x^\alpha$ in some open neighbourhood $U$ of $m$ one may arrange that $X^a = \partial/\partial t$ and that $\tilde{g}$ is the local product

$$d\tilde{s}^2 = \epsilon dt^2 + h_{\alpha\beta} dx^\alpha dx^\beta \tag{35}$$

where $\epsilon(= \pm 1)$ is the sign of $g(X, X)$. The condition $\nabla \tilde{X} = \sigma g$ shows that, on $U$, $\rho$ and $\sigma$ depend only on $t$. Thus $g = \rho^2 \tilde{g}$ and a change of the first coordinate from $t$ to $u$, where $du/dt = \rho$, shows that $\sigma = d\rho/du$ and reveals (34). This establishes part (i) and part (ii) is immediate from part (i). $\square$

This result leads naturally to the next theorem which is similar to a result in [6, 7]. The proof of this theorem gives a nice example of a situation where, unlike many of those in the previous section, non-trivial solutions of the Sinyukov equation (15) arise and how the inversion of the corresponding pair $(a, \lambda)$ to get the pair $(g', \psi)$ is carried out. This will be useful later this section.

**Theorem 6.** *Let $(M, g)$ be a space-time and $X$ a global vector field on $M$ with exactly the properties described in the general condition of lemma 4. Then the metric $g$ is one of a family of (locally) projectively related metrics on $U$ whose general member is given by $g' = \kappa[Fg - cF^2\tilde{X} \otimes \tilde{X}]$ where $F = (1 + \epsilon c\rho^2)^{-1}$, $\kappa$ and $c$ are constants satisfying the restrictions $\kappa > 0$ and $\epsilon c > 0$ (and so $F$ is a positive function on $M$) and, as in lemma 4, $\epsilon = \pm 1$ is the sign of $g(X, X)$. In the coordinates of lemma 4(i), $g'$ is given by*

$$ds'^2 = \kappa\{\epsilon F^2 du^2 + F\rho^2 h_{\alpha\beta} dx^\alpha dx^\beta\} \tag{36}$$

**Proof.**

First note that, under the conditions of the theorem, the tensors $a$ and $\lambda$ defined on $M$ by $a = c_1 g + c_2 \tilde{X} \otimes \tilde{X}$ and $\lambda = (c_2 d\rho/du)\tilde{X}$, where $c_1$ and $c_2$ are constants satisfying $c_1 > 0 < \epsilon c_2$, together satisfy (15). Then $a$ is non-degenerate and $\lambda$ is an exact 1-form, on $M$. (It is remarked here that this collection of solutions of (15) for the pair $(a, \lambda)$ is not claimed to be the general solution of (15) and thus the family of (locally) projectively related metrics on $U$ claimed in the theorem is not claimed to be the complete such family.) In appropriate co-ordinates $g$ is given by $\epsilon du^2 + \rho^2 dS^2$ (Lemma 4) and in these co-ordinates $a$ is $c_1 \left[\epsilon(1 + \epsilon c\rho^2)du^2 + \rho^2 dS^2\right]$ (since $\tilde{X} = \epsilon\rho du$), where $dS^2 = h_{\alpha\beta} dx^\alpha dx^\beta$ and $c \equiv c_2/c_1$, $\epsilon c > 0$. Now (Lemma 1) $F = |\det g/\det a| = (1 + \epsilon c\rho^2)^{-1}$ and $a^{-1}$, inverse to $a$, on $M$ is easily seen to be $\kappa[\epsilon F du^2 + \rho^2 dS^2]$ or $a^{-1} = \kappa[g - cF\tilde{X} \otimes \tilde{X}]$, where $0 < \kappa \equiv 1/c_1$. Then $g' = Fa^{-1}$ (Lemma 1) gives the result claimed in the theorem.□

Now consider the general situation when the curvature class of $(M, g)$ is **C** (and so $M$ is non-flat). The only possibility, for this curvature class, of a local or global metric $g'$ non-trivially ($\nabla \neq \nabla'$) projectively related to $g$, wherever both are defined, is when $(M, g)$ has holonomy type $R_{15}$ (theorem 4(ii)) and when $\lambda$ does not vanish identically on $M$. So, suppose $m \in M$ with $\lambda(m) \neq 0$ so that $\lambda$ *is nowhere zero over some connected, open neighbourhood $V$ of $m$.* (This is where the assumption that the holonomy type is $R_{15}$ is used.) Suppose, as one can from the curvature class **C** condition, that $V$ is chosen so that it admits a nowhere-zero smooth ([10], p262) vector field $k$ such that $R^a{}_{bcd}k^d = 0$ on $V$ (and with $k(m)$ the unique non-zero solution of this equation at $m \in V$ up to a scaling). Then the conditions of lemma 2 apply and so, on $V$, $\lambda_{a;b} = cg_{ab}$ with $c$ constant (thus $\lambda$ is homothetic) and from (19) $\lambda_a = \eta k_a$ for some smooth nowhere-zero function $\eta$ on $V$ and, also from (19), and theorem 1(ii),

$$a_{ab} = \phi g_{ab} + \gamma\lambda_a\lambda_b \tag{37}$$

for smooth functions $\phi$ and $\gamma$ on $V$. Now substitute (37) into (15) and, for any $m' \in V$ and for $v \in T_{m'}M$ which is non-null and ($g$-)orthogonal to $\lambda$ at $m'$ contract with $v^a v^b$ to see that $\phi_{,a}(m') = 0$ and so, since $V$ is connected, $\phi$ is constant on $V$. A back substitution and contraction with $v^a$ (recalling that $\lambda$ is nowhere zero on $V$) reveals that $\gamma c = 1$ on $V$ and hence that $\gamma = \frac{1}{c}$ is a non-zero constant (and $c \neq 0$). Thus $(\lambda^a\lambda_a)_{,b} = 2c\lambda_b$ is nowhere zero on $V$ from which it follows that $\lambda$, and hence the vector field $k$ which annihilates *Riem* on $V$ is non-null over some open dense subset of $V$. Suppose now that $V$ is adjusted (if necessary) so that $\lambda$ is nowhere zero or null on $V$. Then lemma 4(ii) applies for $\tilde{X} = \lambda$ (after a linear adjustment in the coordinate $u$) and a family of (non-trivially) projectively related metrics is given by (36). In this case Lemma 2(c) and theorem 1(ii) ensure that (37) is the only possible form for $a$, and so the constancy of $\phi$ and $\gamma$ that follows from substitution into (15) means that (36) gives *all* metrics $g'$ (non-trivially) locally projectively related (on $V$) to $g$.

With a choice of timelike co-ordinate $u = t$ (so that $\epsilon = -1$) and $h$ as a 3-metric of constant curvature, lemma 4 and theorem 6 apply to the FRWL cosmological metrics. The FRWL cosmological metrics thus provide examples of non-trivially locally projectively related pairs of metrics whose common holonomy type is $R_{15}$ and whose curvature class is, in general, **A**. However, these metrics also contain examples where the curvature is of class **C** (and holonomy type $R_{15}$) and are special cases of the metrics described in theorem 5 of [4]. (The Einstein static universe is of holonomy type $R_{13}$ and curvature class **C** and generates no non-trivial projectively related metrics (theorem 4(ii) or 3(ii)).

Finally, as examples of spacetimes relevant to this section, but where $g'$ of the form (36) is *not* the general form of the projectively related metrics, consider the spacetime metrics $ds^2 = 2du\,dv + u^2d\sigma^2$ (which has

holonomy type $R_{11}$) and $ds^2 = \epsilon_1 dt^2 + \epsilon_2 dz^2 + z^2 d\sigma^2$ (which has holonomy type $R_{10}$ or $R_{13}$, depending on choice of $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$) where $d\sigma^2 = h_{\alpha\beta}(x^3, x^4)dx^\alpha dx^\beta$, each taken as defined on some open subset of $\mathbb{R}^4$ and with the restrictions $u > 0$ and $z > 0$, respectively. These metrics arise (as equations (7.10) and (7.19)) in [23] as generic examples of curvature class **D** spacetimes having non-trivial geodesically equivalent metrics. These spacetimes each admit one global covariantly constant (co)vector field $Y$ and it can be shown that each also admits a global homothetic gradient (co)vector field $X$, so that $Y_{a;b} = 0$ and $X_{a;b} = g_{ab}$. The geodesically equivalent metrics $(g', \psi)$ may be derived in each case from the simple construction $a = c_1 g + c_2 Y \otimes Y + c_3 X \otimes X + c_4(X \otimes Y + Y \otimes X), \lambda = c_3 X + c_4 Y$ where $c_1, c_2, c_3$ and $c_4$ are all real constants.

## 6.2. The holonomy types $R_9$ and $R_{14}$

Now suppose that $(M, g)$ is a *non-flat* space-time *of holonomy type $R_9$ or $R_{14}$*. Then $M$ may be disjointly decomposed in an obvious notation as $M = A \cup \text{int}B \cup \text{int}C \cup \text{int}D \cup F$ (since $\text{int}O$ is empty) where [10] $A$, $A \cup B$, $A \cup B \cup C$ and $A \cup B \cup C \cup D$ are open (and the last of these is also dense) in $M$. Thus $F$ is closed and $M \setminus F$ is open and dense in $M$. Now suppose that $U$ is a non-empty, connected, open subset of $M$ on which is defined a Lorentz metric $g'$ with Levi-Civita connection $\nabla'$ and which is projectively related to $g$ on $U$. Then one has the disjoint decompositions $U = A' \cup B' \cup C' \cup D' \cup O' = A' \cup \text{int}B' \cup \text{int}C' \cup \text{int}D' \cup F'$ where $A' = A \cap U$, $B' = B \cap U$ etc (and $\text{int}A' = (\text{int}A) \cap U = A'$, $\text{int}B' = (\text{int}B) \cap U$, etc) $A'$ open in $U$ and $\text{int}F' = \text{int}(F \cap U) = \emptyset$. Thus $U \setminus F'$ is open and dense in $U$. If the holonomy type of $(M, g)$ is $R_{14}$ and any of the subsets $\text{int}B'$ and $\text{int}C'$ are non-empty then, noting that the holonomy algebras associated with the (necessarily open) components of $\text{int}B'$ and $\text{int}C'$ are subalgebras of the $R_{14}$ holonomy algebra of dimension $\geq 2$ and hence of type $R_6, R_8, R_9, R_{11}, R_{12}$ or $R_{14}$, (for $\text{int}C'$) and $R_7$ (for $\text{int}B'$), it follows from theorem 4(ii) and (iii) applied these components that $\nabla = \nabla'$ on $\text{int}B' \cup \text{int}C'$. The situation in $\text{int}D'$ was described at the end of the last section. If the holonomy type of $(M, g)$ is $R_9$, $B = \emptyset$ and a similar argument using theorems 4(i) and 4(ii) shows that $\nabla = \nabla'$ on $\text{int}C' \cup \text{int}D'$. So interest is directed towards the (assumed non-empty) open subset $A'$ of $U$.

If $W$ is any non-empty, open, connected subset of $A'$ and hence of $U$ it follows from table 1 and the condition $W \subset A'$ that if $(M, g)$ has holonomy type $R_9$ then so also does $(W, g)$ whilst if $(M, g)$ has holonomy type $R_{14}$, $(W, g)$ has holonomy type $R_9, R_{12}$ or $R_{14}$. If the metrics $g$ and $g'$ are projectively related on $U$ and if the associated 1-form $\lambda$ vanishes on $W$, (15) shows that $\nabla a = 0$ on $W$. Since $W \subset A'$ it follows from theorem 1(iv) and the remarks following it that $a = \alpha g$ on $W$ ($\alpha \in \mathbb{R}$). Theorem 3(iii) then shows that the trivial solution $\lambda = 0$ and $a = \alpha g$ on $U$ is the only solution of (15) on $U$ and hence $\lambda = 0$ and $a = \alpha g$ is the only solution of (15) on $M$. So suppose that $\lambda$ does not vanish on any non-empty, open subset of $A'$. Then the subset $A'' \equiv A' \setminus \{m \in A' : \lambda(m) = 0\}$ is open and dense in $A'$ and open in $U$. Now, with $(M, g)$ of holonomy type $R_9$ or $R_{14}$, for any non-empty open subset $W \subset A''$, $(W, g)$ has holonomy type $R_9$ or $R_{14}$ (since, if it were $R_{12}$, $\lambda \equiv 0$ on $W$ from [23] or theorem 3(i)) and so each $m \in A''$ admits a connected, open neighbourhood $V \subset A''$ on which is defined a nowhere-zero ($\nabla$-)recurrent null vector field $l$ which is unique up to a nowhere-zero scaling. The recurrence condition on $l$ shows that $l$ is hypersurface-orthogonal on $V$ (so with all index movements done using the metric $g$, $l_{[a;b}l_{c]} = 0$) and so $V$ may be chosen so that a (nowhere-zero) rescaled version of $l$ (also denoted $l$) exists on it and which is (recurrent and) normal, satisfying $l_a = u_{,a}$ and, from the recurrence condition, $l_{a;b} = \beta l_a l_b$ for functions $u$ and $\beta$ on $V$. The set $V$ may be chosen so that the restriction of $l$ to $V$ may be augmented into a smooth null tetrad $l, n, x, y$ on $V$. Then the Ricci identity for $l$ gives on $V$ (again using a comma for a partial derivative)

$$(l_{a;b} = \beta l_a l_b) \Rightarrow l_d R^d{}_{abc} = l_{a;bc} - l_{a;cb} = l_a F_{bc} \qquad (F_{ab} = 2l_{[a}\beta_{,b]}) \qquad (38)$$

Since there are no non-trivial solutions for $k$ of the equation $k^a R_{abcd} = 0$ at any point of $A$ and hence of $A''$, $F$ and hence $d\beta$ are nowhere zero on $V$. Now, from (38), $l \wedge x$ and $l \wedge y$ lie in the kernel of the map $f$ in (1) at each point of $V$ and so from lemma 2, $l, x$ and $y$ are eigenvectors of $\nabla\lambda$ with equal eigenvalues. Thus

$$\lambda_{ab} \equiv \lambda_{a;b} = \rho g_{ab} + \sigma l_a l_b \qquad (39)$$

18

holds on $V$ for functions $\rho$ and $\sigma$ which are smooth on $V$ since $\lambda_{a;b}n^an^b$ and $\lambda^a{}_a$ are. Then using (39) in (18) one gets

$$a_{ae}R^e{}_{bcd} + a_{be}R^e{}_{acd} = \sigma(g_{ac}l_bl_d + g_{bc}l_al_d - g_{ad}l_bl_c - g_{bd}l_al_c) \tag{40}$$

A contraction of (40) with $l^a$ (defining $L_a = a_{ab}l^b$ and noting that it is nowhere-zero on $V$ since $a$ is non-degenerate on $V$) and use of (38) gives $L_e R^e{}_{bcd} = L_b F_{cd}$ on $V$. Then a contraction of this equation first with $L^b$ shows that $L$ is null and second with $l^b$ (and a comparison with (38) contracted with $L^a$) shows that $L^e l_e F_{ab} = 0$ and hence that $L_a l^a = 0$, on $V$. It follows that $L_a$ and $l_a$ are proportional on $V$ and so

$$a_{ab}l^b = \phi l_a \tag{41}$$

for some clearly smooth function $\phi$ on $V$ which is nowhere zero on $V$ because of the non-degeneracy of $a$.

Since $l = du$ on $V$, a differentiation of (41) and use of (15) (and $l_{a;b} = \beta l_a l_b$) gives

$$g_{ab}(\lambda_c l^c) + \lambda_a l_b = l_a \phi_{,b} \qquad (\Rightarrow \quad \lambda_a = \phi_{,a} = \phi' l_a) \tag{42}$$

where the last steps follow from a rank argument and the fact that $l$ is nowhere zero on $V$ (to get $\lambda_a l_b = l_a \phi_{,b}$) and the fact that $\phi$ is a (nowhere-zero) function of $u$ only (and a prime denotes $d/du$). Thus $\phi'$ is a smooth nowhere-zero function of $u$ on $V$. So the function $\rho$ in (39) vanishes on $V$. It is then noted, from section 4, that $\phi$, being a potential of $\lambda$, satisfies $\phi + c' = \frac{1}{2}a_{ab}g^{ab}$ for some $c' \in \mathbb{R}$. It follows that $\lambda$ may be taken as representative of the recurrent null direction on $A''$ and one may take $l = \lambda$ on each such neighbourhood $V$. Then the function $\beta$ may be redefined as that associated with $\lambda$ ($\lambda_{a;b} = \beta \lambda_a \lambda_b$) on each such $V$. Now $\beta$ is defined on $A''$ and it easily follows from (38) that if $\beta$ vanishes over any non-empty open subset of $A''$ the Ricci identity contradicts the curvature class $\mathbf{A}$ condition on that set. Hence, for later convenience, one may remove the set of zeros of $\beta$ from $A''$ leaving behind an open dense subset $A'''$ of $A''$ (and hence of $A'$) on which $\beta$ is nowhere-zero. It also follows from (42) that, with this identification of $l$ and $\lambda$, that $\phi$ may, up to the choice of additive constant, be chosen as the co-ordinate $u$. This will be done at a convenient point later in the argument

Continuing the argument one has $\lambda_{a;b} = \beta \lambda_a \lambda_b$ on $V$ and so, from (41), $a_{ab}l^b = \phi l_b$. Now contract (40) with $l^c$ and use (38) to get

$$\phi(l_a\beta_b + \beta_a l_b) - (l_a a_{be}\beta^e + a_{ae}\beta^e l_b) = 2\beta l_a l_b \tag{43}$$

where $\beta^a = g^{ae}\beta_e$ and $\beta_a \equiv \beta_{,a}$. A contraction of (43) with $n^b$ and then another with $n^an^b$ and some algebraic simplifications then yields

$$a_{ab}\beta^b = \phi\beta_a - \beta l_a \tag{44}$$

Now the choice $l = \lambda$ means that $l$ and $\beta$ are defined, smooth and nowhere-zero on $A'''$. So consider the function $\beta_{,a}l^a$ and note that it cannot vanish over any non-empty open subset $\bar{V}$ of $A'''$. For if it did then $(\beta_{,a}l^a)_{;b} = 0$ on $\bar{V}$ and so, by the recurrence of $l$, $l^a\beta_{a;b} = 0$ on $\bar{V}$. But then a differentiation of (44) followed by a contraction with $\beta^a$ and use of (15) and (44) (noting that, on $\bar{V}$, $\beta^a\lambda_a = \beta^a l_a = 0$ from (42) and that, from (44) $a_{ab}\beta^a\beta^b_{;c} = \phi\beta_b\beta^b_{;c}$) shows that $\beta_a$ is null in addition to satisfying $\beta_a l^a = 0$ on $\bar{V}$. Thus $\beta_a$ is proportional to $l_a$ and the bivector $F$ in (38) vanishes on $\bar{V}$. This contradiction completes the argument. So a further reduction to an open dense subset $\tilde{A}$ of $A'''$ (and hence of $A'$) will be made by omitting those points where $\beta_{,a}l^a$ vanishes. Henceforth, one works on $\tilde{A}$.

It follows that the bivector $F$ is timelike over $\tilde{A}$ and one may choose a connected, open neighbourhood $V \subset \tilde{A}$ about any $m \in \tilde{A}$ and a smooth null tetrad $l, n, x, y$ on $V$ so that, on $V$,

$$\beta_a = pn_a + ql_a \tag{45}$$

for smooth functions $p$ and $q$ on $V$ with $p$ ($= \beta_a l^a$) nowhere zero on $V$ and $F$ is now the bivector $2pl_{[a}n_{b]}$. On substituting (45) into (44) one finds

$$a_{ab}n^b = \phi n_a + \xi l_a, \qquad \xi \equiv -\beta/p \tag{46}$$

19

so that $\xi$ is smooth and nowhere-zero on $V$ (since $\beta$ is). Equations (41) and (46) reveal that the 2-space, $l \wedge n$, is, at each $m \in V$, a timelike *invariant* 2-space of $a$ and so it follows that the orthogonal complement, $x \wedge y$, of $l \wedge n$ is also (see [10]). Since this latter 2-space is spacelike it follows that, at each $m \in V$, the spacelike vectors $x$ and $y$ may be adjusted so that they are eigenvectors of $a$ at $m$ [10]. However, such an adjustment may not lead to *smooth* eigenvector fields of $a$ over some open neighbourhood of $m$ [25]. To consider this problem further let a smooth second order symmetric tensor field $K$ be defined on $V$ in the original smooth (unadjusted) tetrad $l, n, x, y$ by

$$K_{ab} = a_{ab} - \phi(l_a n_b + n_a l_b) - \xi l_a l_b \tag{47}$$

Then $K_{ab}l^b = K_{ab}n^b = 0$ on $V$ and $x \wedge y$ is clearly an invariant 2-space of $K$ at each $m \in V$ and which contains an orthogonal pair of eigenvectors $\tilde{x}$ and $\tilde{y}$ of $K$ but which may not lead to smooth eigenvector fields of $K$ on $V$ (but the 2-spaces, $x \wedge y$ and $\tilde{x} \wedge \tilde{y}$ coincide). Using the tetrad $l, n, \tilde{x}, \tilde{y}$ at $m$ one has $K_{ab}\tilde{x}^b = \tilde{C}g_{ab}\tilde{x}^b \equiv \tilde{C}\tilde{x}_a$ and similarly, $K_{ab}\tilde{y}^b = \tilde{D}\tilde{y}_a$ for $\tilde{C}, \tilde{D} \in R$. Then, from (47) $a_{ab}\tilde{x}^b = \tilde{C}\tilde{x}_a$ and $a_{ab}\tilde{y}^b = \tilde{D}\tilde{y}_a$ at $m$ and $\tilde{C} \neq 0 \neq \tilde{D}$ since $a$ is non-degenerate. It follows that the Segre type of $K$ at a given point of $V$ is either $\{(1,1)11\}$ (if $\tilde{C} \neq \tilde{D}$ at that point) or $\{(1,1)(11)\}$ (if $\tilde{C} = \tilde{D}$ at that point) and no further degeneracies are permitted (since the repeated eigenvalue in each 2-space, $l \wedge n$, is zero) [10]. Since this is true for each such set $V$, it is true over $\tilde{A}$ and so $\tilde{A}$ may be written as the disjoint union of two (not necessarily non-empty) sets, upon each (non-empty) one of which the Segre type of $K$ (including degeneracies) is constant. The member $A_1$ of this pair of sets corresponding to Segre type $\{(1,1)11\}$ is then open in $\tilde{A}$ (and hence in $A'$) since the solutions (eigenvalues) of the characteristic polynomial of $a$ depend smoothly on the (smooth) coefficients of this polynomial. If the set corresponding to Segre type $\{(1,1)(11)\}$ is denoted by $A_2$ then $A_1 \cup \text{int} A_2$ is an *open dense* subset of $\tilde{A}$ and hence of the original $A'$ and all eigenvalues of $K$ and hence of $a$ are smooth on $A_1$ and $\text{int} A_2$ since the Segre type is constant on each of them. The associated eigenvector fields may then be chosen locally smoothly (in some simply connected open neighbourhood of any point of $A_1$ and $\text{int} A_2$) [25]. Thus one has an open dense subset of $A'$ any point of which admits a neighbourhood with all properties so far derived for such sets and additionally on which a null tetrad $(l, n, x, y)$ may be chosen with $l$ and $n$ as before, so that (47) still holds and $K_{ab} = Cx_a x_b + Dy_a y_b$ with *smooth* eigenvector fields now labelled $x$ and $y$ and *smooth* eigenvalues now labelled $C$ and $D$ and with $C$ and $D$ everywhere distinct (respectively, everywhere equal) if the Segre type of $K$ is $\{(1,1)11\}$ (respectively $\{(1,1)(11)\}$). Thus, on any such neighbourhood, (47) gives $a_{ab}x^b = Cx_a$ and $a_{ab}y^b = Dy_a$.

Henceforth, attention will be directed to the two open sets $A_1$ and $\text{int} A_2$. These open sets can be related to the holonomy structure and can be described (briefly) in the following way. Consider the range space of the map $f$ in (1) at $m \in \tilde{A}$. It is clear from (38) that if a bivector $G$ is in the range of $f$ at $m$ (so that for some bivector $H$, $G_{ab} = R_{abcd}H^{cd}$ at $m$) then $G_{ab}l^b = \mu l_a$ for $\mu \in \mathbb{R}$. In terms of the null tetrad fields constructed just before (45) this means that the range space of $f$ at $m$ is spanned by the bivectors $l \wedge n$, $l \wedge x$, $l \wedge y$ and $x \wedge y$ (subject to consistency with the curvature class **A** condition). Now, recalling the Ambrose-Singer theorem (see the end of section 3), the holonomy algebra of $(W, g)$ for some connected open neighbourhood $W \subset \tilde{A}$ of $m$ can be constructed by first choosing (any) $m \in W$ and, for each $m' \in U$ and each curve $c$ from $m'$ to $m$ in $W$, computing the range space of $f$ at $m'$ and parallely transporting each member of it to $m$ along $c$. If this is done for each such $m'$ and $c$, the collection of bivectors accumulated at $m$ spans the holonomy algebra of $(W, g)$. Suppose that the algebra thus arising contains a member which does *not* satisfy the condition that it is a simple bivector whose blade contains $l$ (and this must be the case if the holonomy type of $(W, g)$ is $R_{14}$). Now because $l$ is recurrent, it gives rise, by parallel transport, to a 1-dimensional (null) holonomy invariant distribution on $W$. Thus any bivector in the range space of $f$ at some point of $W$ will, if it is simple with $l$ in its blade, remain that way under parallel transport. It follows that the range space of $f$ at some $m' \in U$ must contain a bivector which does not have the property that it is simple with $l$ in its blade. From this it follows that $R_{abcd}x^{[c}y^{d]} \neq 0$ at $m'$ and hence in some connected open neighbourhood of $m'$. If all bivectors in the holonomy algebra of $(W, g)$ (and hence in the range space of $f$ at any $m' \in U$) are simple with $l$ in their blade the holonomy group of $(W, g)$ is $R_9$ and $R_{abcd}x^{[c}y^{d]} = 0$ at each point of $W$.

Now contract (40) with $x^a y^b$ and use the results $a_{ab}x^b = Cx_a$ and $a_{ab}y^b = Dy_a$ to get, at any point of $\tilde{A}$

$$(C - D)R_{abcd}x^{[c}y^{d]} = 0 \tag{48}$$

Thus if the holonomy group of $(W, g)$ is of type $R_{14}$ the above work together with (48) shows that $W$ must intersect $A_2$ (and, in fact, int$A_2$) and hence that $m$ lies in the *closure* of $A_2$. Let $m \in A_1 \cup \text{int}A_2$ and let $V$ be a connected, open neighbourhood of $m$ with the properties given to such neighbourhoods just before (45) and which is contained in exactly one of these two (disjoint) sets. Then $C$ and $D$ may be taken as smooth eigenvalues of $K$ on $V$ with associated smooth eigenvectors $x$ and $y$. If the holonomy type of $(V, g)$ is $R_{14}$, $V \subset A_2$ and $C = D$ on $V$, whilst if the holonomy type of $(V, g)$ is $R_9$, one could have either $V \subset A_2$ and $C = D$ on $V$ or $V \subset A_1$ and $C \neq D$ on $V$. Considering the holonomy type $R_{14}$ first, (47) gives

$$a_{ab} = \phi(l_a n_b + n_a l_b) + \xi l_a l_b + C(x_a x_b + y_a y_b) \tag{49}$$
$$= Cg_{ab} + \xi l_a l_b + (\phi - C)(l_a n_b + n_a l_b)$$

where the completeness relation $g_{ab} = l_a n_b + n_a l_b + x_a x_b + y_a y_b$ has been employed. Since $\frac{1}{2}a_{ab}g^{ab} = (\phi + C)$ and $\phi$ and $u$ are each potentials for $\lambda = l$, $C$ *is constant on* $V$. Then if we choose to adapt coordinates to suit this particular $(a, \lambda)$ solution one can choose the zero of coordinate $u$ such that $\phi = u + C$, and (49) becomes

$$a_{ab} = u(l_a n_b + n_a l_b) + \xi l_a l_b + Cg_{ab} \tag{50}$$

If the holonomy group is of type $R_9$ then there is no control over $C$ and $D$ and one may write from (47) (to cover either of the possibilities)

$$a_{ab} = u(l_a n_b + n_a l_b) + \xi l_a l_b + C(x_a x_b + y_a y_b) + E y_a y_b \tag{51}$$

where $E \equiv D - C$ is smooth on $V$.

With $V$ as above, suppose that the holonomy type of $(V, g)$ is $R_{14}$ so that (50) holds with $C$ constant. Then a substitution of (50) into (15) (using $l_{a;b} = \beta l_a l_b$ and $n^a n_{a;b} = 0$) and a contraction first with $n^a n^b$ gives

$$2n_a - 2\beta \xi l_a = \xi_{,a} \tag{52}$$

and then with $n^a$ gives

$$un_{a;b} = (g_{ab} - l_a n_b - n_a l_b) - u\beta n_a l_b \tag{53}$$

Then (53) gives an expression for $n_{a;b}$ from which it is clear from the non-degeneracy of $g$ that $u$ cannot vanish on $V$ and easily checked that $n_{[a;b}n_{c]} = 0$ on $V$ and so $n$ is *hypersurface-orthogonal* on $V$. Thus reducing $V$, if necessary, some nowhere-zero scaling $\tilde{n}$ of $n$ satisfies $\tilde{n} = dv$ for some smooth function $v$ on $V$. Then from (53) and the recurrence of $l$ it is easily seen that $l$ and $n$ are involutive and so span a 2-dimensional smooth distribution $D_1$ on $V$. Also the subsets of $V$ of constant $u$ *and* $v$ span a smooth 2-dimensional distribution $D_2$ on $V$. Now, for $m \in V$, $T_m M = D_1(m) \oplus D_2(m)$ and so by considering the local flat charts for $D_1$ and $D_2$, respectively, one may choose $V$ and a coordinate system $y^a$ on $V$ such that $\{\partial/\partial y^1, \partial/\partial y^2\}$ span $D_1$ and $\{\partial/\partial y^3, \partial/\partial y^4\}$ span $D_2$, on $V$ (see e.g. [26] chapter 6, page 28). One can then check by reintroducing the functions $u$ and $v$ and relabelling $y$ by $x$ that $V$ may be chosen as a coordinate domain with coordinate functions $u$, $v$, $x^3$, $x^4$ such that the metric $g$ takes the form

$$ds^2 = 2Pdudv + g_{\alpha\beta}dx^\alpha dx^\beta \tag{54}$$

where $P$ is a smooth function on $V$ and Greek letters take the values 3, and 4. [Thus in this coordinate system, $l_a$ has components $(1, 0, 0, 0)$, $l^a = (0, P^{-1}, 0, 0)$, $n_a = (0, P, 0, 0)$ and $n^a = (1, 0, 0, 0)$.] On writing out the recurrence condition on $l$ in these coordinates as $l_c \Gamma^c_{ab} = -\beta l_a l_b$ one easily finds that $P$ is independent of $x$ and $y$ and that the $g_{\alpha\beta}$ are independent of $v$. Then (53) in these coordinates (and with $a = \alpha, b = \beta$)

gives $-un_c\Gamma^c_{\alpha\beta} = g_{\alpha\beta}$ which is $-uP\Gamma^2_{\alpha\beta} = g_{\alpha\beta}$ and which easily leads to $u\partial g_{\alpha\beta}/\partial u = 2g_{\alpha\beta}$. Thus, after possibly a change of coordinates $x^3$ and $x^4$ (and possibly also of the open subset $V$) the metric becomes

$$ds^2 = 2P(u,v)dudv + u^2 e^{2w(x^3,x^4)}((dx^3)^2 + (dx^4)^2) \tag{55}$$

for some smooth function $w$ on $V$. Further remarks must be added here regarding the function $w$ and this is, perhaps, best done after the holonomy type $R_9$ has been considered.

Suppose now that the holonomy type of $(V,g)$ is $R_9$. Then either (50) or (51) may hold on $V$ with the nowhere-zero functions $C$ and $D$ and the vector fields $x$ and $y$ smooth on $V$ and if it is (50) one again arrives at (55). If (51) holds with $C$ and $E$ smooth, distinct and nowhere-zero on $V$ then $C = a_{ab}x^ax^b$ and $C + E = a_{ab}y^ay^b$ and differentiating, using (15), $l = \lambda$ and $x_ax^a_{;b} = y_ay^a_{;b} = 0$, one easily finds that $C$ *and $E$ are distinct non-zero constants on $V$*. Now write $a_{ab}x^b = Cx_a$, differentiate and contract with $y^a$ using using (15) and (51). One finds that $Ey^ax_{a;b} = 0$. Similarly one can get $Ex^ay_{a;b} = 0$ It follows that $y^ax_{a;b} = 0$ and $x^ay_{a;b} = 0$ on $V$. Since $l^ax_{a;b} = l^ay_{a;b} = 0$ from the constancy of $l^ax_a$ and $l^ay_a$, respectively, and the recurrence condition on $l$, one gets $x_{a;b} = l_ar_b$ and $y_{a;b} = l_as_b$ for smooth 1-form fields $r$ and $s$ on $V$. It is then easily checked, from a back substitution, that $x_a = -ur_a$ and $y_a = (E-u)s_a$ and so, since then $(E-u)$ cannot vanish on $V$,

$$x_{a;b} = -u^{-1}l_ax_b \qquad y_{a;b} = (E-u)^{-1}l_ay_b \tag{56}$$

holds on $V$. Equation (56) shows that the vector fields $x$ and $y$ are hypersurface orthogonal; $x_{[a;b}x_{c]} = 0$, $y_{[a;b}y_{c]} = 0$. The calculation for $n_{a;b}$ which produced (53) in the case $C = D$ goes through in a similar way to before, and is now

$$un_{a;b} = (g_{ab} - l_an_b - n_al_b) - u\beta n_al_b + (\frac{E}{u-E})y_ay_b \tag{57}$$

However, the consequence $n_{[a;b}n_{c]} = 0$ still holds and, again, $n$ is hypersurface orthogonal. One thus again arrives at the metric (54). But now the hypersurface orthogonality of $x$ and $y$ means that $V$ may be chosen so that one may write $x_a = hq_{,a}$ and $y_a = h'q'_{,a}$ for smooth functions $h$, $h'$, $q$ and $q'$ on $V$. Then using the coordinates $u,v,x^3(= q)$ and $x^4(= q')$, write out (57) with the indices $a$ and $b$ set equal to $\alpha$ and $\beta$ as before, to get $-un_c\Gamma^c_{\alpha\beta} = g_{\alpha\beta} + E(u-E)^{-1}y_\alpha y_\beta$. The only new information is yielded by the choices $\alpha = \beta = 3$ and $\alpha = \beta = 4$ and is $g_{33} = G_{33}(x^3,x^4)u^2$ and $g_{44} = G_{44}(x^3,x^4)(u-E)^2$ for smooth functions $G_{33}$ and $G_{44}$ on $V$. A back substitution into either of (56) then shows that $G_{33}$ is independent of $x^4$ and that $G_{44}$ is independent of $x^3$. Thus the functions $G_{33}$ and $G_{44}$ may be absorbed into the coordinates and in these new coordinates, $x$ and $y$, on open subset $V$ the metric becomes

$$ds^2 = 2P(u,v)dudv + u^2dx^2 + (u-E)^2dy^2 \tag{58}$$

For either metric (55) or (58) the final piece of information, that is, the nature of the function $P(u,v)$, may be deduced from (52). First note from (46) and (52) that the function $\xi$ is a nowhere-zero function of $u$ and $v$ only, on $V$. Then the recurrence condition on $l$ and either (55) or (58) give $\beta = -P^{-1}\partial P/\partial u$ from which (52) gives

$$\frac{\partial\xi}{\partial u} = 2\xi\frac{1}{P}\frac{\partial P}{\partial u} \qquad \frac{\partial\xi}{\partial v} = 2P \qquad \frac{\partial\xi}{\partial x} = 0 \qquad \frac{\partial\xi}{\partial y} = 0 \tag{59}$$

The first equation in (59) gives $P = \sqrt{\xi}\frac{df}{dv}$ where $f$ is some arbitrary function of only $v$. The second equation in (59) then becomes $\frac{\partial\xi}{\partial v} = 2\sqrt{\xi}\frac{df}{dv}$, whose solution is $\xi = (f(v) + B(u))^2$ where $B$ is some arbitrary function of only $u$. Thus we have $\frac{\partial\xi}{\partial u} = 2\sqrt{\xi}\frac{dB}{du}$. Now note that, from (59), $2Pdudv = \frac{\partial\xi}{\partial v}dudv$. A change of coordinates from $(u,v)$ to $(u,z = \frac{1}{2}\xi)$ gives $2Pdudv = 2dudz - 4\frac{\partial\xi}{\partial u}du^2$ and hence that, with $b(u)$ an arbitrary function, $2Pdudv$ may be expressed in the form $2dudz + \sqrt{z}b(u)du^2$.

In summary, if $(V,g)$ has holonomy type $R_{14}$, $C(= D)$ is a non-zero constant and the metric takes the form

$$ds^2 = 2dudz + \sqrt{z}b(u)du^2 + u^2 e^{2w(x^3,x^4)}((dx^3)^2 + (dx^4)^2) \tag{60}$$

for some function $w$ whereas if $(V, g)$ has holonomy type $R_9$ the metric takes either the form (60) or

$$ds^2 = 2dudz + \sqrt{z}b(u)du^2 + u^2dx^2 + (u-E)^2dy^2. \tag{61}$$

To distinguish between the holonomy types for the metric (60), one may calculate that the extra bivectors required in the range of the map $f$ in (1) (and alluded to just before (48)) to exclude the $R_9$ case will be zero on $V$ if the function $w$ satisfies the harmonic condition $\partial^2 w/\partial(x^3)^2 + \partial^2 w/\partial(x^4)^2 = 0$ on $V$. Thus the metric (60) is of holonomy type $R_9$ if and only if $w$ is harmonic over $V$.

The general form for the solution pairs $(a, \lambda)$ of (15) and hence of solution pairs $(g', \psi)$ of (11) can now be found. For (60) one may calculate that $n_a = (\frac{1}{2}\sqrt{z}b(u), 1, 0, 0)$ and that $\xi = 2z$. It can be shown that [27] because of a uniqueness in the direction of $\lambda$ the only freedom in $a$ is that of a constant scaling and the addition of a constant multiple of the metric. Hence the most general form of $a$ may be written in the form

$$a_{ab} = \kappa(cu(l_a n_b + n_a l_b) + 2czl_a l_b + g_{ab}), \qquad \lambda_a = c\kappa l_a \tag{62}$$

where $c$ and $\kappa > 0$ are constants, and with $u$ restricted in the chart domain to ensure that $a$ is non-degenerate, i.e. that $1 + cu \neq 0$. Inverting one finds

$$F = e^{2\chi} = \frac{1}{\kappa^4(1 + cu)^2} \tag{63}$$

and then after calculating $a^{-1}$, $g' = Fa^{-1}$ is of the form

$$\frac{1}{\kappa^5}\left(\frac{2dudz}{(1+cu)^3} + \frac{(\sqrt{z}b(u)(1+cu) + 2cz)du^2}{(1+cu)^4} + \frac{u^2 e^{2w(x^3, x^4)}((dx^3)^2 + (dx^4)^2)}{(1+cu)^2}\right) \tag{64}$$

The associated 1-form is $\psi = d(\frac{1}{2}\ln F)$, given by

$$\psi_a = -\frac{c}{1+cu}l_a \tag{65}$$

It is possible to express (64) in a simplified form (identical in form to $g$) through a co-ordinate transformation from $u$ and $z$ to $U$ and $Z$ where $(u = U/(1-cU), z = Z/(1-cU))$ to obtain, with $B(U) \equiv b(U/(1-cU))/(1-cU)$

$$\frac{1}{\kappa^5}\left(2dUdZ + \sqrt{Z}B(U)dU^2 + U^2 e^{2w(x^3, x^4)}((dx^3)^2 + (dx^4)^2)\right) \tag{66}$$

And similarly, for (61)

$$a_{ab} = \kappa(cu(l_a n_b + n_a l_b) + 2czl_a l_b + g_{ab} + cEy_a y_b), \qquad \lambda_a = c\kappa l_a \tag{67}$$

where $c$, $E$ and $\kappa > 0$ are constants, and with $u$ restricted in the chart domain to ensure $1 + cu \neq 0$, and with $c$ and $E$ chosen such that $1 + cE \neq 0$, so that $a$ is non-degenerate. Inverting one finds,

$$F = e^{2\chi} = \frac{1}{\kappa^4(1+cu)^2(1+cE)} \tag{68}$$

and again, after calculating $a^{-1}$, $g' = Fa^{-1}$ takes the form

$$\frac{1}{\kappa^5(1+cE)}\left(\frac{2dudz}{(1+cu)^3} + \frac{(\sqrt{z}b(u)(1+cu) - 2cz)du^2}{(1+cu)^4} + \frac{u^2dx^2}{(1+cu)^2} + \frac{(u-E)^2dy^2}{(1+cE)(1+cu)^2}\right) \tag{69}$$

The projective 1-form $\psi$ has the from given in (65). As in the previous case a co-ordinate transformation, in this case $(u = \mu U/(1 - c\mu U), z = \mu Z/(1 - c\mu U), y = Y/\mu)$, allows (69) to be expressed in a simplified form

$$\frac{1}{\kappa^5}\left(2dUdZ + \sqrt{Z}B(U)dU^2 + U^2dx^2 + (U-E')^2 dY^2\right) \tag{70}$$

where $\mu = \sqrt{1 + cE}$, $B(U) \equiv 1/(1 - c\mu U)b\,(\mu U/(1 - c\mu U))$ and $E' = E\mu^{-3}$.

Thus if $(M, g)$ is non-flat and locally projectively related over a non-empty connected open subset $U$ to $(U, g')$ then, in the previous notation, if $(M, g)$ is of holonomy type $R_9$ each point of an open dense subset of $U$ admits an open neighbourhood $V$ on which either $\nabla' = \nabla$ or $g$ and $g'$ satisfy (60) and (64) (with $w$ harmonic on $V$) or (61) and (69) whereas if $(M, g)$ is of holonomy type $R_{14}$ each point of an open dense subset of $U$ admits an open neighbourhood $V$ on which either $\nabla' = \nabla$ or $g$ and $g'$ satisfy the conditions mentioned earlier for the holonomy type $R_{11}$ case (as dealt with in [23]) or $g$ and $g'$ satisfy (60) and (64) .

## 7. Acknowledgements

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